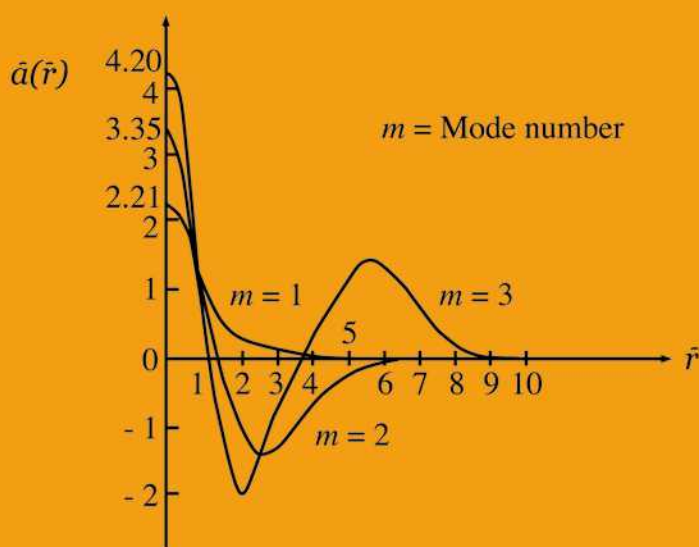


# Nonlinear Optics

Theory, Numerical Modeling,  
and Applications



Partha P. Banerjee

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**Theory, Numerical Modeling,  
and Applications**

**Partha P. Banerjee**

*University of Dayton  
Dayton, Ohio, U.S.A.*



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## Preface

*Nonlinear Optics: Theory, Numerical Modeling, and Applications* is a self-explanatory book in a rather new and changing area, and is geared toward advanced senior or first-year graduate students in electrical engineering and physics. It is assumed that the students taking the course have had exposure to Fourier optics and electro-optics. This book is the culmination of a course on nonlinear optics that I have taught several times at the graduate level over the last ten years, and has also introduced some of the topics of senior-level classes on laser systems. It is also based on my research in the area over the last 20 years.

The unique features of the book are as follows. Students are first reacquainted with pertinent topics from linear optics that are useful in understanding some of the concepts used later on in the book. Thereafter, rigorous treatment of nonlinear optics is developed alongside a heuristic treatment to enable the reader to understand the underlying essential physics, instead of being overwhelmed with extensive tensor calculus. Recent topics of interests, applications, and measurement and calculation techniques are discussed. While the plane wave approach to harmonic generation is first explained, more recent developments such as the effect of beam profile on second harmonic generation, second generation during guided wave propagation, and the combined role of quadratic and cubic nonlinearities are also examined. Cubic nonlinearities are discussed at length along with their effects such as self-focusing and defocusing, self-bending of beams, and spatial solitons. The role of cascaded second-order nonlinearities is also examined. The  $z$ -scan technique and its modification are described in detail as a means of character-

ization of optical nonlinearities. We also discuss other cubic nonlinearity effects such as soliton propagation through nonlinear fibers, with some attention to the recent development of dispersion management in nonlinear optical fibers. Optical bistability and switching in a nonlinear ring cavity as well as during optical propagation across a linear/nonlinear interface are treated at length. Traditional topics such as stimulated Brillouin and Raman scattering are summarized. Also, phase conjugation in a cubically nonlinear material and dynamic holography are introduced. A simple  $k$ -space picture is used to explain phase conjugation of beams and pulses. Thereafter, the nonlinear optics of photorefractive materials is discussed in detail, including applications to dynamic holography, two-wave mixing, phase conjugation, and image processing. Photorefractive crystals as well as organic thin-film photorefractive materials are discussed. Examples of image processing such as edge enhancement using these materials are introduced. The nonlinear optics of liquid crystals is discussed at length, including the effects of applied electric and optical fields (including beams) on the nonlinearity. The effective nonlinearity of liquid crystals is determined from a careful evaluation of the position-dependent nonlinearity in the material. Self-organization plays a vital role in human behavioral system, in the brain, in fluid mechanics, in chemical reactions, etc.—in any system that has nonlinearity and feedback. It is therefore not unnatural to expect self-organization in optical systems as well. In this book, we therefore discuss spatiotemporal effects in nonlinear optical materials, leading to self-organization and spatial pattern formation, using photorefractives as a representative nonlinear medium. Innovative potential applications of self-organization are also presented. Finally, we treat photonic crystals or photonic bandgap structures that can be engineered to yield specific stop-bands for propagating waves, and demonstrate their application in optical bistability and hysteresis, soliton formation, and phase matching during second harmonic generation. Pertinent numerical methods, often used to analyze beam and pulse propagation in nonlinear materials, such as the split-step beam propagation method and the fully adaptive wavelet transform technique, are presented in the Appendices. Also, illustrative problems at the end of each of chapter are intended to aid the student in grasping the fundamentals and applying them to other interesting problems in nonlinear optics. In short, the book extends the concepts of nonlinear optics to areas of recent interest and, in a sense, brings contemporary and ongoing research areas not usually covered in many nonlinear optics books to the attention of readers.

The emphasis of this book is on the understanding of physical principles and potential applications. Students interested in further in-depth coverage of basics are referred to more comprehensive treatments such as the *Handbook of Nonlinear Optics* (Richard L. Sutherland, ed., Marcel Dekker, 2003).

I would like to thank Ms. Cheryl McKay from the University of Alabama in Huntsville for typing parts of the manuscript, my graduate student Ms. Jia Zhang for assistance with most of the figures, all of my graduate students whose work appears in the text who have worked with me through the years, and several students who took the course during the preparation of the manuscript for their helpful comments. Finally, I would like to thank my family and friends for their moral support.

*Partha P. Banerjee*



# Contents

|                                                                                  |            |
|----------------------------------------------------------------------------------|------------|
| <i>Preface</i>                                                                   | <i>iii</i> |
| 1 Optical Propagation in Linear Media                                            | 1          |
| 1.1 Maxwell's Equations                                                          | 1          |
| 1.2 Linear Wave Propagation in Homogeneous, Linear Isotropic Media               | 4          |
| 1.3 Wave Propagation in Anisotropic Media                                        | 12         |
| 1.4 Diffraction                                                                  | 25         |
| 1.5 Dispersion                                                                   | 31         |
| 1.6 Problems                                                                     | 34         |
| References                                                                       | 35         |
| 2 The Origin and Modeling of Optical Nonlinearity                                | 37         |
| 2.1 A Simple Physical Model for Optical Nonlinearity                             | 38         |
| 2.2 Physical Effects of Nonlinear Polarization                                   | 42         |
| 2.3 Mathematical Modeling of Optical Nonlinearities                              | 43         |
| 2.4 An Alternative Approach                                                      | 54         |
| 2.5 Summary                                                                      | 56         |
| 2.6 Problems                                                                     | 57         |
| References                                                                       | 57         |
| 3 Second Harmonic Generation                                                     | 59         |
| 3.1 The Physics of Second Harmonic Generation                                    | 59         |
| 3.2 SHG in Crystals—Mathematical Formulation                                     | 62         |
| 3.3 Phase Matching in Anisotropic Crystal                                        | 70         |
| 3.4 Nonlinear Transverse Effects in Second Harmonic Generation                   | 71         |
| 3.5 SHG in a Medium with Second-Order and Third-Order Nonlinear Susceptibilities | 73         |
|                                                                                  | <i>vii</i> |

|     |                                                                    |     |
|-----|--------------------------------------------------------------------|-----|
| 3.6 | Summary                                                            | 76  |
| 3.7 | Problems                                                           | 77  |
|     | References                                                         | 77  |
| 4   | Self-Refractive of Optical Beams                                   | 79  |
| 4.1 | The Nonlinear Refractive Index Coefficient $n_2$                   | 80  |
| 4.2 | Self-Refractive of Gaussian Beams                                  | 81  |
| 4.3 | Focused Gaussian Beam in a Kerr-Slice: The $z$ -Scan Method        | 90  |
| 4.4 | Self-Bending of Optical Beams                                      | 96  |
| 4.5 | Self-Phase Modulation Due to $\chi^{(2)}$ : Effective $\chi^{(3)}$ | 97  |
| 4.6 | Summary                                                            | 98  |
| 4.7 | Problems                                                           | 99  |
|     | References                                                         | 100 |
| 5   | Optical Bistability                                                | 101 |
| 5.1 | The Nonlinear Ring Cavity                                          | 101 |
| 5.2 | Transverse Optical Bistability                                     | 109 |
| 5.3 | Linear–Nonlinear Interface                                         | 110 |
| 5.4 | Conclusion                                                         | 119 |
| 5.5 | Problems                                                           | 120 |
|     | References                                                         | 120 |
| 6   | Optical Phase Conjugation                                          | 123 |
| 6.1 | Comparison with Holography                                         | 124 |
| 6.2 | The $\mathbf{k}$ -Space Formalism                                  | 125 |
| 6.3 | Semiclassical Analysis: CW Case                                    | 132 |
| 6.4 | Phase Conjugation of Pulses and Transient Phase Conjugate Response | 137 |
| 6.5 | Discussion                                                         | 142 |
| 6.6 | Problems                                                           | 142 |
|     | References                                                         | 144 |
| 7   | Stimulated Brillouin and Raman Scattering                          | 145 |
| 7.1 | Stimulated Brillouin Scattering                                    | 145 |
| 7.2 | Stimulated Raman Scattering                                        | 151 |
| 7.3 | Problems                                                           | 153 |
|     | References                                                         | 154 |
| 8   | Solitons in Optical Fibers                                         | 155 |
| 8.1 | Background on Linear Optical Fibers                                | 156 |
| 8.2 | Fiber Dispersion and Nonlinearity                                  | 158 |

|      |                                                                                                          |     |
|------|----------------------------------------------------------------------------------------------------------|-----|
| 8.3  | Fiber-Optic Solitons and the NLS Equation                                                                | 161 |
| 8.4  | Dispersion Managed Soliton Communication Systems                                                         | 164 |
| 8.5  | Dark Solitons                                                                                            | 166 |
| 8.6  | Optical Shocks and Self-Steepening of Pulses                                                             | 171 |
| 8.7  | Problems                                                                                                 | 173 |
|      | References                                                                                               | 173 |
| 9    | Photorefractive Nonlinear Optics                                                                         | 175 |
| 9.1  | Introduction                                                                                             | 175 |
| 9.2  | The Kukhtarev Equations and Their Simplification                                                         | 177 |
| 9.3  | Beam Fanning and Distortion in Photorefractive Materials                                                 | 181 |
| 9.4  | Two-Wave Mixing in Photorefractive Materials                                                             | 194 |
| 9.5  | Four-Wave Mixing and Phase Conjugation in Photorefractive Materials                                      | 196 |
| 9.6  | Self-Phase Conjugation and Edge Enhancement                                                              | 200 |
| 9.7  | Organic Photorefractive Materials                                                                        | 202 |
| 9.8  | Problems                                                                                                 | 206 |
|      | References                                                                                               | 207 |
| 10   | Nonlinear Optical Properties of Nematic Liquid Crystals                                                  | 209 |
| 10.1 | The Liquid Crystalline State of Matter                                                                   | 210 |
| 10.2 | Classification of Liquid Crystals                                                                        | 210 |
| 10.3 | Liquid Crystal Alignment                                                                                 | 211 |
| 10.4 | Principles of the Continuum Theory                                                                       | 213 |
| 10.5 | Director Distribution of Homogeneously Aligned Nematic Liquid Crystal Under an External Electrical Field | 215 |
| 10.6 | Nonlinear Optical Properties from Optically Induced Molecular Reorientation                              | 224 |
| 10.7 | Optically Induced Reorientational Nonlinearity with External Voltage                                     | 228 |
| 10.8 | Analysis of Beam Propagation in Liquid Crystals and $z$ -Scan                                            | 235 |
| 10.9 | Problems                                                                                                 | 240 |
|      | References                                                                                               | 241 |
| 11   | Self-Organization in Nonlinear Optical Systems                                                           | 243 |
| 11.1 | Introduction                                                                                             | 243 |
| 11.2 | Nonlinear Two-Dimensional Systems with Feedback                                                          | 245 |

|                   |                                                                                         |     |
|-------------------|-----------------------------------------------------------------------------------------|-----|
| 11.3              | Self-Organization in Photorefractive Materials                                          | 249 |
| 11.4              | Theory of Self-Organization in Photorefractive Materials                                | 255 |
| 11.5              | Instability Criterion and the Dispersion Relation                                       | 257 |
| 11.6              | Nonlinear Eigenmodes in the Steady State                                                | 259 |
| 11.7              | Model of Hexagonal Formation Based on Transverse Electrical Instability                 | 264 |
| 11.8              | Potential Applications                                                                  | 265 |
| 11.9              | Problems                                                                                | 267 |
|                   | References                                                                              | 268 |
| 12                | Nonlinear Optics of Photonic Bandgap Structures                                         | 271 |
| 12.1              | Introduction                                                                            | 271 |
| 12.2              | The Linear Dispersion Relation                                                          | 272 |
| 12.3              | The Effect of Cubic Nonlinearity                                                        | 276 |
| 12.4              | Second Harmonic Generation in Photonic Bandgap Structures                               | 282 |
| 12.5              | Problems                                                                                | 287 |
|                   | References                                                                              | 288 |
| <i>Appendix A</i> | <i>The Split Step Beam Propagation Method</i>                                           | 289 |
|                   | <i>A.1 Examples</i>                                                                     | 290 |
|                   | <i>References</i>                                                                       | 294 |
| <i>Appendix B</i> | <i>Wavelet Transforms and Application to Solution of Partial Differential Equations</i> | 295 |
|                   | <i>B.1 Introduction to Wavelets</i>                                                     | 295 |
|                   | <i>B.2 Wavelet Properties and Scaling Functions</i>                                     | 296 |
|                   | <i>B.3 Digital Filters and Multi-Resolution Analysis</i>                                | 298 |
|                   | <i>B.4 Application of Wavelet Transform to the NLS Equation</i>                         | 302 |
|                   | <i>References</i>                                                                       | 306 |
|                   | <i>Index</i>                                                                            | 307 |

# 1

## Optical Propagation in Linear Media

In this chapter, we will review some of the properties of optical waves propagating through an unbounded linear medium. We believe that this review will serve as an adequate foundation for the topics in nonlinear optics, to which the entire book is devoted. To this end, we enunciate Maxwell's equations and derive the wave equation in a linear homogeneous isotropic medium. We define intrinsic impedance, the Poynting vector and irradiance, as well as introduce the concept of polarization. We then expose readers to concepts of plane-wave propagation through anisotropic media, introduce the index ellipsoid, and show an application of electro-optic materials. We also summarize concepts of Fresnel and Fraunhofer diffraction, starting from the paraxial wave equation, and examine the linear propagation of a Gaussian beam. Finally, we expose readers to the important topic of dispersion, which governs the spreading of pulses during propagation in a medium. More importantly, we show how by knowing the dispersion relation, one can deduce the underlying partial differential equation that needs to be solved to find the pulse shapes during propagation. We hope this chapter presents readers with most of the background material required for starting on the rigors of nonlinear optics, which will be formally introduced in Chap. 2. For further reading on related topics, the reader is referred to Cheng (1983), Banerjee and Poon (1991), Goodman (1996), Yariv (1997), and Poon and Banerjee (2001).

### 1 MAXWELL'S EQUATIONS

In the study of optics, we are concerned with four vector quantities called electromagnetic fields: the electric field strength  $\mathbf{E}$  (V/m); the electric flux density  $\mathbf{D}$  (C/m<sup>2</sup>); the magnetic field strength  $\mathbf{H}$  (A/m); and the magnetic flux

density  $\mathbf{B}$  (Wb/m<sup>2</sup>). The fundamental theory of electromagnetic fields is based on Maxwell's equations. In differential form, these are expressed as

$$\nabla \cdot \mathbf{D} = \rho, \quad (1-1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1-2)$$

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \quad (1-3)$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \mathbf{J}_C + \frac{\partial \mathbf{D}}{\partial t}, \quad (1-4)$$

where  $\mathbf{J}$  is the current density (A/m<sup>2</sup>) and  $\rho$  denotes the electric charge density (C/m<sup>3</sup>).  $\mathbf{J}_C$  and  $\rho$  are the sources generating the electromagnetic fields.

We can summarize the physical interpretation of Maxwell's equations as follows: Equation (1-1) is the differential representation of *Gauss' law for electric fields*. To convert this to an integral form, which is more physically transparent, we integrate Eq. (1-1) over a volume  $V$  bounded by a surface  $S$  and use the *divergence theorem* (or *Gauss' theorem*),

$$\int_V \nabla \cdot \mathbf{D} dV = \oint_S \mathbf{D} \cdot d\mathbf{S} \quad (1-5)$$

to obtain

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV. \quad (1-6)$$

This states that the electric flux  $\oint_S \mathbf{D} \cdot d\mathbf{S}$  flowing out of a surface  $S$  enclosing  $V$  equals the total charge enclosed in the volume.

Equation (1-2) is the magnetic analog of Eq. (1-1) and can be converted to an integral form similar to Eq. (1-6) by using the divergence theorem once again:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0. \quad (1-7)$$

The right-hand sides (RHSs) of Eqs. (1-2) and (1-7) are zero because, in the classical sense, magnetic monopoles do not exist. Thus the magnetic flux is always conserved.

Equation (1-3) enunciates *Faraday's law of induction*. To convert this to an integral form, we integrate over an open surface  $S$  bounded by a line  $C$  and use Stokes' theorem,

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_C \mathbf{E} \cdot d\boldsymbol{\ell}, \quad (1-8)$$

to obtain

$$\int_C \mathbf{E} \cdot d\boldsymbol{\ell} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad (1-9)$$

This states that the electromotive force (emf)  $\oint_C \mathbf{E} \cdot d\ell$  induced in a loop is equal to the time rate of change of the magnetic flux passing through the area of the loop. The emf is induced in a sense such that it opposes the variation of the magnetic field, as indicated by the minus sign in Eq. (1-9); this is known as *Lenz's law*.

Analogously, the integral form of Eq. (1-4) reads

$$\oint_C \mathbf{H} \cdot d\ell = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} + \int_S \mathbf{J}_C \cdot d\mathbf{S}, \quad (1-10)$$

which states that the line integral of  $\mathbf{H}$  around a closed loop  $C$  equals the total current (conduction and displacement) passing through the surface of the loop. When first formulated by Ampere, Eqs (1-4) and (1-10) only had the conduction current term  $\mathbf{J}_C$  on the RHS. Maxwell proposed the addition of the displacement current term  $\partial \mathbf{D} / \partial t$  to include the effect of currents flowing through, for instance, a capacitor.

For a given current and charge density distribution, note that there are four equations [Eqs. (1-1)–Eqs. (1-4)] and, at first sight, four unknowns that need to be determined to solve a given electromagnetic problem. As such, the problem appears well posed. However, a closer examination reveals that Eqs. (1-3) and (1-4), which are vector equations, are really equivalent to six scalar equations. Also, by virtue of the *continuity equation*,

$$\nabla \cdot \mathbf{J}_C + \frac{\partial \rho}{\partial t} = 0, \quad (1-11)$$

Equation (1-1) is not independent of Eq. (1-4) and, similarly, Eq. (1-2) is a consequence of Eq. (1-3). We can verify this by taking the divergence on both sides of Eqs. (1-3) and (1-4) and by using the continuity equation [Eq. (1-11)] and a vector relation,

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad (1-12)$$

to simplify. The upshot of this discussion is that, strictly speaking, there are six independent scalar equations and twelve unknowns (viz., the  $x$ ,  $y$ , and  $z$  components of  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{H}$ , and  $\mathbf{B}$ ) to solve for. The six more scalar equations required are provided by the constitutive relations,

$$\mathbf{D} = \epsilon \mathbf{E}, \quad (1-13a)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad (1-13b)$$

where  $\epsilon$  denotes the permittivity (F/m) and  $\mu$  is the permeability (H/m) of the medium. Note that we have written  $\epsilon$  and  $\mu$  as scalar constants. This is true for a linear, homogeneous, isotropic medium. A medium is *linear* if its properties do not depend on the amplitude of the fields in the medium. It is *homogeneous* if its properties are not functions of space. Furthermore, the

medium is *isotropic* if its properties are the same in all direction from any given point. For now, we will assume the medium to be linear, homogeneous, and isotropic. However, anisotropic materials will be studied when we examine electro-optic effects in Section 3.

Returning our focus to linear, homogeneous, isotropic media, constants worth remembering are the values of  $\varepsilon$  and  $\mu$  for free space or vacuum:  $\varepsilon_0 = (1/36\pi) \times 10^{-9}$  F/m and  $\mu_0 = 4\pi \times 10^{-7}$  H/m. For dielectrics, the value of  $\varepsilon$  is greater than  $\varepsilon_0$ , and contains a material part characterized by a dipole moment density  $\mathbf{P}$  (C/m<sup>2</sup>).  $\mathbf{P}$  is related to the electric field  $\mathbf{E}$  as

$$\mathbf{P} = \chi\varepsilon_0\mathbf{E}, \quad (1-14)$$

where  $\chi$  is the electric susceptibility and indicates the ability of the electric dipoles in the dielectric to align themselves with the electric field. The  $\mathbf{D}$  field is the sum of  $\varepsilon_0\mathbf{E}$  and  $\mathbf{P}$ ,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(1 + \chi)\mathbf{E} \triangleq \varepsilon_0\varepsilon_r\mathbf{E}, \quad (1-15)$$

where  $\varepsilon_r$  is the relative permittivity, so that

$$\varepsilon = (1 + \chi)\varepsilon_0. \quad (1-16)$$

Similarly, for magnetic materials,  $\mu$  is greater than  $\mu_0$ .

## 2 LINEAR WAVE PROPAGATION IN HOMOGENEOUS, LINEAR ISOTROPIC MEDIA

In this section, we first derive the wave equation and review some of the traveling wave-type solutions of the equation in different coordinate systems. We define the concept of intrinsic impedance, Poynting vector and intensity, and introduce the subject of polarization.

### 2.1 Traveling-Wave Solutions

In Section 1, we enunciated Maxwell's equations and the constitutive relations. For a given  $\mathbf{J}_C$  and  $\rho$ , we remarked that we could, in fact, solve for the components of the electric field  $\mathbf{E}$ . In this subsection, we see how this can be performed. We derive the wave equation describing the propagation of the electric and magnetic fields and find its general solutions in different coordinate systems. By taking the curl of both sides of Eq. (1-3) we have

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}), \quad (2-1)$$

where we have used the second of the constitutive relations [Eq. (1-13a)] and assumed  $\mu$  to be space- and time-independent. Now employing Eq. (1-4), Eq. (2-1) becomes

$$\nabla \times \nabla \times \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu \frac{\partial \mathbf{J}_C}{\partial t}, \quad (2-2)$$

where we have used the first of the constitutive relations [Eq. (1-13a)] and assumed  $\epsilon$  to be time-independent. Then, by using the vector relationship

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad \nabla^2 = \nabla \cdot \nabla, \quad (2-3)$$

in Eq. (2-2), we obtain

$$\nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{J}_C}{\partial t} + \nabla(\nabla \cdot \mathbf{E}). \quad (2-4)$$

If we now assume the permittivity  $\epsilon$  to be space-independent as well, then we can recast the first of Maxwell's equations [Eq. (1-1)] in the form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \quad (2-5)$$

using the first of the constitutive relations [Eq. (1-13a)]. Incorporating Eq. (2-5) into Eq. (2-4), we finally obtain

$$\nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}_C}{\partial t} + \frac{1}{\epsilon} \nabla \rho, \quad (2-6)$$

which is a wave equation having source terms on the RHS. In fact, Eq. (2-6), being a vector equation, is really equivalent to three scalar equations, one for every component of  $\mathbf{E}$ . Expressions for the Laplacian ( $\nabla^2$ ) operator in Cartesian ( $x, y, z$ ), cylindrical ( $r, \theta, z$ ), and spherical ( $R, \theta, \phi$ ) coordinates are given as follows:

$$\nabla_{\text{rect}}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad (2-7)$$

$$\nabla_{\text{cyl}}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad (2-8)$$

$$\nabla_{\text{sph}}^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial}{\partial \theta}. \quad (2-9)$$

In space free of all sources ( $\mathbf{J}_C = 0, \rho = 0$ ), Eq. (2-6) reduces to the homogeneous wave equation:

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (2-10)$$

A similar equation may be derived for the magnetic field  $\mathbf{H}$ ,

$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}. \quad (2-11)$$

We caution readers that the  $\nabla^2$  operator, as written in Eqs. (2-7)–Eqs. (2-9), must be applied only after decomposing Eqs. (2-10) and (2-11) into scalar equations for three orthogonal components in  $\mathbf{a}_x$ ,  $\mathbf{a}_y$ , and  $\mathbf{a}_z$ . However, for the rectangular coordinate case only, these scalar equations may be recombined and interpreted as the Laplacian  $\nabla_{\text{rect}}^2$  acting on the total vector.

Note that the quantity  $\mu\epsilon$  has the units of  $(1/\text{velocity})^2$ . We call this velocity  $v$  and define it as

$$v^2 = \frac{1}{\mu\epsilon}. \quad (2-12)$$

For free space,  $\mu = \mu_0$ ,  $\epsilon = \epsilon_0$ , and  $v = c$ . We can calculate the value of  $c$  from the values of  $\epsilon_0$  and  $\mu_0$  mentioned in Section 1. This works out to  $3 \times 10^8$  m/sec.

Let us now examine the solutions of equations of the type of Eq. (2-10) or (2-11) in different coordinate systems. For simplicity, we will analyze the homogeneous wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \nabla^2 \psi = 0, \quad (2-13)$$

where  $\psi$  may represent a component of the electric field  $\mathbf{E}$  or of the magnetic field  $\mathbf{H}$  and where  $v$  is the velocity of the wave.

In Cartesian coordinates, the general solution is

$$\begin{aligned} \psi(x, y, z, t) = & c_1 f(\omega_0 t - k_{0x}x - k_{0y}y - k_{0z}z) \\ & + c_2 g(\omega_0 t - k_{0x}x + k_{0y}y - k_{0z}z) \\ & c_1, c_2 \text{ constants,} \end{aligned} \quad (2-14)$$

and with the condition

$$\frac{\omega_0^2}{k_{0x}^2 + k_{0y}^2 + k_{0z}^2} \triangleq \frac{\omega_0^2}{k_0^2} = v^2. \quad (2-15)$$

In Eq. (2-15),  $\omega_0$  is the (*angular*) frequency (rad/sec) of the wave and  $k_0$  is the *propagation constant* (rad/m) in the medium. Because the ratio  $\omega_0/k_0$  is a constant, the medium of propagation is said to be *nondispersive*. We can reexpress Eq. (2-14) as

$$\psi(x, y, z, t) = c_1 f(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{R}) + c_2 g(\omega_0 t + \mathbf{k}_0 \cdot \mathbf{R}), \quad (2-16)$$

where

$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z, \quad (2-17a)$$

$$\mathbf{k}_0 \triangleq k_{0x}\mathbf{a}_x + k_{0y}\mathbf{a}_y + k_{0z}\mathbf{a}_z, \quad (2-17b)$$

$\mathbf{k}_0$  is called the *propagation vector* and  $|\mathbf{k}_0| = k_0$ ;  $\mathbf{a}_x$ ,  $\mathbf{a}_y$ , and  $\mathbf{a}_z$  denote the unit vectors in the  $x$ ,  $y$ ,  $z$  directions, respectively.

In one spatial dimension (viz.,  $z$ ), the wave equation [Eq. (2-13)] reads

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (2-18)$$

and its general solution is

$$\psi(z, t) = c_1 f(\omega_0 t - k_0 z) + c_2 g(\omega_0 t + k_0 z), \quad v = \frac{\omega_0}{k_0}. \quad (2-19)$$

Note that Eq. (2-14) or (2-16) comprises the superposition of two waves, traveling in opposite directions. We can define a wave as a disturbance of some form, characterized by a recognizable velocity of propagation. Although this definition sounds rather loose at the moment, we will see that it perfectly describes all the different types of waves that we will encounter in this book. Let us now consider a special case:  $c_1 \neq 0$ ,  $c_2 = 0$ . Observe that if  $\psi$  is a constant, so is  $\omega_0 t - \mathbf{k}_0 \cdot \mathbf{R}$ . Hence,

$$\mathbf{k}_0 \cdot \mathbf{R} = \omega_0 t + \text{constant}. \quad (2-20)$$

But this is the equation of a plane perpendicular to  $\mathbf{k}_0$  with  $t$  as a parameter; hence the wave is called a *plane wave*. With increasing  $t$ ,  $\mathbf{k}_0 \cdot \mathbf{R}$  must increase so that Eq. (2-20) always holds. For instance, if  $\mathbf{k}_0 = k_0 \mathbf{a}_z$  ( $k_0 > 0$ ) and  $\mathbf{R} = z\mathbf{a}_z$ ,  $z$  must increase as  $t$  increases. This means that the wave propagates in the  $+z$  direction. For  $c_1 = 0$ , we have a plane wave traveling in the opposite direction. The *wavefronts*, defined as the surfaces joining all points of equal phase  $\omega_0 t \pm \mathbf{k}_0 \cdot \mathbf{R}$ , are planar.

Consider now the cylindrical coordinate system. The simplest case is that of cylindrical symmetry, which requires that

$$\psi(r, t) = \frac{C}{r^{1/2}} \exp\{j\omega_0 t \pm k_0 r\}, \quad r > 0 \text{ and } C \text{ constant} \quad (2-21)$$

approximately satisfy the wave equation [Eq. (2-13)]. We remark that the exact solution has a Bessel-function-type dependence on  $r$  if we assume  $\psi$  to be time-harmonic, i.e., of the form  $\psi = \text{Re}[\psi_p \exp j\omega_0 t]$ , where  $\text{Re}[\cdot]$  means “the real part of.”

Finally, we give the solutions of the wave equation in a spherical coordinate system. For spherical symmetry ( $\partial/\partial\phi = 0 = \partial/\partial\theta$ ), the wave equation [Eq. (2-13)] with Eq. (2-9) assumes the form

$$R\left(\frac{\partial^2\psi}{\partial R^2} + \frac{2}{R}\frac{\partial\psi}{\partial R}\right) = \frac{\partial^2(R\psi)}{\partial R^2} = \frac{1}{v^2}\frac{\partial^2(R\psi)}{\partial t^2}. \quad (2-22)$$

Now, Eq. (2-22) is of the same form as Eq. (2-18). Hence using Eq. (2-19), we can write down the solution of Eq. (2-22) as

$$\psi = \frac{c_1}{R}f(\omega_0 t - k_0 \mathbf{R}) + \frac{c_2}{R}g(\omega_0 t - k_0 \mathbf{R}), \quad c_1, c_2 \text{ constants} \quad (2-23)$$

## 2.2 Intrinsic Impedance and the Poynting Vector

So far in our discussion of Maxwell's equations, as well the wave equation and its solutions, we made no comments on the components of  $\mathbf{E}$  and  $\mathbf{H}$ . The solutions of the wave equation [Eq. (2-13)] in different coordinate systems are valid for every component  $\mathbf{E}$  and  $\mathbf{H}$ . We point out here that the solutions of the wave equation previously discussed hold, in general, only in an unbounded medium.

In this subsection, we first show that electromagnetic wave propagation is transverse in nature in an unbounded medium, and derive the relationships between the existing electric and magnetic fields. In this connection, we define the intrinsic or characteristic impedance of a medium, which is similar in concept to the characteristic impedance of a transmission line. We also introduce the concept of power flow during electromagnetic propagation and define the Poynting vector and the irradiance.

In an unbounded isotropic, linear, homogeneous medium free of sources, electromagnetic wave propagation is transverse in nature. This means that the only components of  $\mathbf{E}$  and  $\mathbf{H}$  are those that are transverse to the direction of propagation. To check this, we consider propagating electric and magnetic fields of the forms

$$\begin{aligned} \mathbf{E} = \mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z &\triangleq \text{Re}\{E_{0y}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_x \\ &\quad + E_{0y}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_y \\ &\quad + E_{0z}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_z\}, \end{aligned} \quad (2-24)$$

$$\begin{aligned} \mathbf{H} = \mathbf{H}_x + \mathbf{H}_y + \mathbf{H}_z &\triangleq \text{Re}\{H_{0x}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_x \\ &\quad + H_{0y}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_y \\ &\quad + H_{0z}\exp[j(\omega_0 t - k_0 z)]\mathbf{a}_z\}, \end{aligned} \quad (2-25)$$

Substituting Eq. (2-25) into Eq. (1-1), using Eqs. (1-13a), and with  $\rho=0$ , we find

$$E_{0z} = 0. \quad (2-26)$$

This means that there is no component of the electric field in the direction of propagation. The only possible components of  $\mathbf{E}$  then must be in a plane transverse to the direction of propagation. Similarly, we can show that

$$H_{0z} = 0. \quad (2-27)$$

We comment here that the above results only hold true for an isotropic medium. However in an anisotropic material,  $\mathbf{D}$  is perpendicular to the direction of propagation, as will be shown in Section 3.

Furthermore, substitution of Eqs. (2-24), (2-25) and (2-26) with  $E_{0z} = 0 = H_{0z}$  into the third of Maxwell's equations [Eq. (1-5)],

$$k_0 E_{0y} \mathbf{a}_x - k_0 E_{0x} \mathbf{a}_y = -\mu \omega_0 (H_{0y} \mathbf{a}_x + H_{0x} \mathbf{a}_y).$$

We can then write [using Eqs. (2-12) and (2-15)]

$$H_{0x} = -\frac{1}{\eta} E_{0y} \quad H_{0y} = \frac{1}{\eta} E_{0x}, \quad (2-28)$$

where

$$\eta \triangleq \frac{\omega_0}{k_0} \mu = \left( \frac{\mu}{\epsilon} \right)^{\frac{1}{2}} \quad (2-29)$$

is called the *intrinsic* or *characteristic impedance* of the medium. The characteristic impedance has the units of V/A or  $\Omega$ . Now using Eqs. (2-25)–Eqs. (2-29), we see that

$$\mathbf{E} \cdot \mathbf{H} = 0 \quad (2-30)$$

meaning that the electric and magnetic fields are orthogonal to each other, and that  $\mathbf{E} \times \mathbf{H}$  is along the direction of propagation ( $z$ ) of the electromagnetic field. Similar relationships can be established in other coordinate systems.

Note that  $\mathbf{E} \times \mathbf{H}$  has the units of W/m<sup>2</sup>, reminiscent of power per unit area. All electromagnetic waves carry energy, and for isotropic media, the energy flow occurs in the direction of propagation of the wave. As we shall see in Section 5, this is not true for anisotropic media. The *Poynting vector*  $\mathbf{S}$ , defined as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad (2-31)$$

is a power density vector associated with the electromagnetic field.

In a linear, homogeneous, isotropic unbounded medium, we can choose the electric and magnetic fields to be of the form

$$\begin{aligned}\mathbf{E}(z, t) &= \text{Re}[E_0 \exp[j(\omega_0 t - k_0 z)]] \mathbf{a}_x, \\ \mathbf{H}(z, t) &= \text{Re} \left[ \frac{E_0}{\eta} \exp[j(\omega_0 t - k_0 z)] \right] \mathbf{a}_y,\end{aligned}\quad (2-32)$$

where  $E_0$  is, in general, a complex quantity. This choice is consistent with Eqs. (2-28)–Eqs. (2-30). If we find  $\mathbf{S}$ , note that it is a function of time. Therefore it is more convenient to define the time-averaged power, or *irradiance*  $I$ ,

$$I = \langle |\mathbf{S}| \rangle = \frac{\omega_0}{2\eta} \int_0^{2\pi/\omega_0} |\mathbf{S}| dt = \frac{|E_0|^2}{2\mu} = \epsilon v \frac{|E_0|^2}{2}. \quad (2-33)$$

In the chapters to follow, the irradiance will often be referred to as the *intensity*. Unless otherwise stated, it will always be taken to be proportional to the magnitude squared of the complex electric field.

### 2.3 Polarization

In this subsection we introduce readers to the concept of *polarization* of the electric field. The polarization describes the time-varying behavior of the electric field. The polarization describes the time-varying behavior of the electric field vector at a given point in space. A separate description of the magnetic field is not necessary, because the direction of  $\mathbf{H}$  is related to that of  $\mathbf{E}$ .

Assume, for instance, that in Eq. (2-25),  $E_{0z} = 0$  and

$$E_{0x} = |E_{0x}|, \quad E_{0y} = |E_{0y}| e^{-j\phi_0}, \quad (2-34)$$

where  $\phi_0$  is a constant.

First, consider the case where  $\phi_0 = 0$  or  $\pm\pi$ . Then, the two components of  $\mathbf{E}$  are in phase, and

$$\mathbf{E} = (|E_{0x}| \mathbf{a}_x \pm |E_{0y}| \mathbf{a}_y) \cos(\omega_0 t - k_0 z). \quad (2-35)$$

The direction of  $\mathbf{E}$  is fixed on a plane perpendicular to the direction of propagation (this plane is referred to as the *plane of polarization*) and does not vary with time, and the electric field is said to be *linearly polarized*.

As a second case, assume  $\phi_0 = \pm\pi/2$  and  $|E_{0x}| = |E_{0y}| = E_0$ . In this case, from Eq. (2-25),

$$\mathbf{E} = E_0 \cos(\omega_0 t - k_0 z) \mathbf{a}_x \pm E_0 \sin(\omega_0 t - k_0 z) \mathbf{a}_y. \quad (2-36)$$

When monitored at a certain point  $z = z_0$  during propagation, the direction of  $\mathbf{E}$  is no longer fixed along a line, but varies with time according to  $\theta = \omega_0 t - k_0 z_0$ , where  $\theta$  represents the angle between  $\mathbf{E}$  and the (transverse)  $x$  axis. The amplitude of  $\mathbf{E}$  (which is equal to  $E_0$ ) is, however, still a constant. This is an example of *circular polarization* of the electric field. When  $\phi_0 = -\pi/2$ ,  $E_y$  leads  $E_x$  by  $\pi/2$  [see Eq. (2-36)]. Hence as a function of time,  $\mathbf{E}$  describes a clockwise circle in the  $x-y$  plane as seen head-on at  $z = z_0$ . Similarly, for  $\phi_0 = +\pi/2$ ,  $\mathbf{E}$  describes a counterclockwise circle.

In the general case,

$$\begin{aligned} \mathbf{E} = \mathbf{E}_x + \mathbf{E}_y &\triangleq E_x \mathbf{a}_x + E_y \mathbf{a}_y = |E_{0x}| \cos(\omega_0 t - k_0 z) \mathbf{a}_x \\ &+ |E_{0y}| \cos(\omega_0 t - k_0 z - \phi_0) \mathbf{a}_y. \end{aligned} \quad (2-37)$$

As in the case of circularly polarized waves (where  $|E_{0x}|^2 + |E_{0y}|^2 = E_0^2 = \text{constant}$ ), the direction of  $\mathbf{E}$  is no longer fixed because  $\mathbf{E}_x$  and  $\mathbf{E}_y$  vary with time. We can trace this variation on the  $\mathbf{E}_x$ - $\mathbf{E}_y$  plane by eliminating the harmonic variations of  $(\omega_0 t - k_0 z)$  in Eq. (2-37). To do this we note that

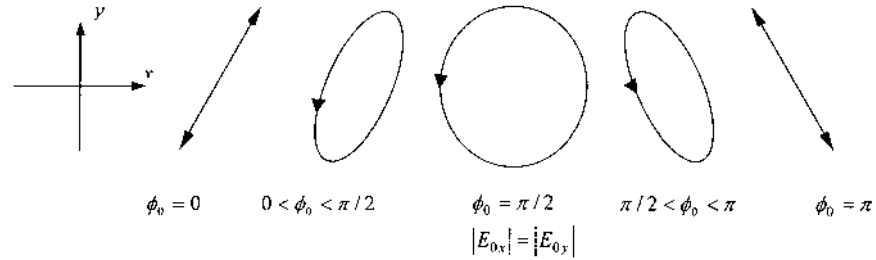
$$\begin{aligned} \frac{E_y}{|E_{0y}|} &= \cos(\omega_0 t - k_0 z) \cos \phi_0 + \sin(\omega_0 t - k_0 z) \sin \phi_0 \\ &= \frac{E_x}{|E_{0x}|} \cos \phi_0 + \left\{ 1 - \left( \frac{E_x}{|E_{0x}|} \right)^2 \right\}^{\frac{1}{2}} \sin \phi_0. \end{aligned} \quad (2-38)$$

After a little algebra, we can reexpress Eq. (2-38) as

$$\left\{ \frac{E_x}{|E_{0x}|} \right\}^2 - 2 \left\{ \frac{E_x}{|E_{0x}|} \right\} \left\{ \frac{E_y}{|E_{0y}|} \right\} \cos \phi_0 + \left\{ \frac{E_y}{|E_{0y}|} \right\}^2 = \sin^2 \phi_0 \quad (2-39)$$

which is the equation of an ellipse; hence the wave is said to be *elliptically polarized*. Note that for values of  $\phi_0$  equal to 0 or  $\pm\pi$  and  $\pm\pi/2$  (with  $|E_{0x}| = |E_y| = E_0$ ), the polarization configurations reduce to the linearly and circularly polarized cases, respectively.

Fig. 1 illustrates various polarization configurations corresponding to different values of  $\phi_0$  to demonstrate clearly linear, circular, and elliptical polarizations. In this figure, we show the direction of rotation of the  $\mathbf{E}$  field vector with time, and its magnitude for various  $\phi_0$ . When  $\phi = 0$  or  $\pm\pi$ , the  $\mathbf{E}$  field is linearly polarized and the  $\mathbf{E}$  vector does not rotate. Unless otherwise stated, we will, throughout the book, assume all electric fields to be linearly polarized and choose the transverse axes such that one of them coincides with the direction of the electric field. Therefore the magnetic field will be along the other transverse axis and will be related to the electric field via the characteristic impedance  $\eta$ .



**Figure 1** Various polarization configurations corresponding to different values of  $\phi_0$ .  $|E_{0x}| \neq |E_{0y}|$  unless otherwise stated.

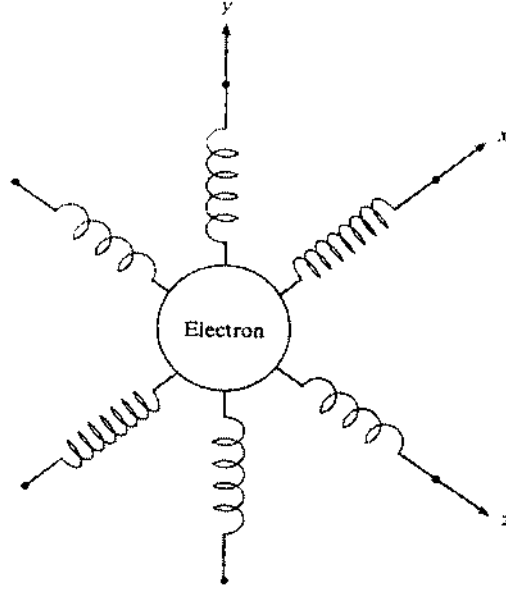
The state of polarization of light traveling in the  $z$  direction can alternately be represented in terms of a complex vector  $(E_{0x} \cdot E_{0y})^T$ . For instance,  $(1 \ 0)^T$ ,  $(0 \ 1)^T$ , and  $(2)^{-\frac{1}{2}} (1 \ 1)^T$ , are examples of linearly polarized light. Such vectors are called *Jones vectors*. Furthermore, the transformation from one polarization state to another may be represented by a matrix called the *Jones matrix*. For instance, transformation from  $(1 \ 0)^T$  to  $(0 \ 1)^T$  may be realized by applying the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . An optical element that can perform this transformation is the *half-wave plate*. This is discussed more in Section 3.

### 3 WAVE PROPAGATION IN ANISOTROPIC MEDIA

Thus far, in this book, we have studied the effects of wave propagation through isotropic media. However, many materials (e.g., crystals) are *anisotropic*. In this section, we will study linear wave propagation in a medium that is homogeneous and magnetically isotropic ( $\mu_0$  constant) but that allows for electrical anisotropy. By this we mean that the polarization produced in the medium by an applied electric field is no longer just a constant times the field, but critically depends on the direction of the applied field in relation to the anisotropy of the medium.

#### 3.1 The Dielectric Tensor

Fig. 2 depicts a model illustrating anisotropic binding of an electron in a crystal. Anisotropy is taken into account by assuming different spring constants in each direction. (In the isotropic case, all spring constants are equal.) Consequently, the displacement of the electron under the influence of an external electric field depends not only on the magnitude of the field but also on its direction. It follows, in general, that the vector  $\mathbf{D}$  will no longer be the direction of  $\mathbf{E}$ . Thus in place of Eq. (1-13a), the components of  $\mathbf{D}$  and  $\mathbf{E}$  are



**Figure 2** Model illustrating anisotropic binding of an electron in a crystal (the spring constants may be different in each direction).

related by the following equation. [Equation (1-13b) still holds true, as we are considering a magnetically isotropic medium.]

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z, \quad (3-1a)$$

$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z, \quad (3-1b)$$

$$D_z = \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z, \quad (3-1c)$$

or,

$$D_i = \sum_{j=1}^3 \epsilon_{ij}E_j, \quad (3-2)$$

where  $i, j = 1$  for  $x$ ,  $2$  for  $y$ ,  $3$  for  $z$ . Equations (3-1a)–(3-1c), or Eq. (3-2), are customarily written as

$$\mathbf{D} = \epsilon \epsilon \mathbf{E} \quad (3-3)$$

where  $\mathbf{D} = [D_1 \ D_2 \ D_3]^T$ ,  $\mathbf{E} = [E_1 \ E_2 \ E_3]^T$ , and

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \quad (3-4)$$

or simply as

$$D_i = \varepsilon_{ij} E_j. \quad (3-5)$$

The  $3 \times 3$  matrix in Eq. (3-4) is commonly known as the *dielectric tensor*. Equation (3-5) is merely a shorthand representation of Eq. (3-2), using the *Einstein convention*. The Einstein convention assumes an implied summation over repeated indices (viz.,  $j$ ) on the same side (RHS or LHS) of an equation.

In a lossless medium, the dielectric tensor is symmetric; that is,

$$\varepsilon_{ij} = \varepsilon_{ji} \quad (3-6)$$

and has only six independent elements [see, for instance, Haus (1984) for proof].

Now, it is well known that any real symmetric matrix can be diagonalized through a coordinate transformation. Hence the dielectric tensor can assume the diagonal form

$$\varepsilon = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \quad (3-7)$$

The new coordinate system is called the *principal axis system*, the three  $\varepsilon$ 's are known as the *principal dielectric constants*, and the Cartesian coordinate axes are called the *principal axes*. Three crystal classes (shown in Table 1) can be identified in terms of Eq. (3-7): *cubic*, *uniaxial*, and *biaxial*. Because most of the crystals used for electro-optic devices are uniaxial, we will therefore concentrate only on these types of crystals in our subsequent discussions. Note that for uniaxial crystals, the axis, which is characterized by a component of  $\varepsilon = \varepsilon_z$ , is called the *optic axis*. When  $\varepsilon_z > \varepsilon_x = \varepsilon_y$ , the crystal is *positive uniaxial*, and when  $\varepsilon_z < \varepsilon_x = \varepsilon_y$ , it is *negative uniaxial*. A word on notation: We will use the subscripts 1, 2, 3 and  $x, y, z$  interchangeably throughout the text.

**Table 1** Crystal Classes and Some Common Examples

| Principal axis system | Cubic                      | Uniaxial                                                                                                                                          | Biaxial       |
|-----------------------|----------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|---------------|
| Common examples       | Sodium chloride<br>Diamond | Quartz<br>(positive, $\varepsilon_x = \varepsilon_y < \varepsilon_z$ )<br>Calcite<br>(negative, $\varepsilon_x = \varepsilon_y > \varepsilon_z$ ) | Mica<br>Topaz |

### 3.2 Plane-Wave Propagation in Uniaxial Crystals; Birefringence

To advance a consolidated treatment of plane-wave propagation in uniaxial crystals, it is advantageous to first describe, in general, the expression for  $\mathbf{D}$  in terms of  $\mathbf{E}$ . This can be achieved by rewriting Eqs. (1-3) and (1-4) and realizing that the operators  $\partial/\partial t$  and  $\nabla$  may be replaced according to

$$\frac{\partial}{\partial t} \rightarrow j\omega_0; \nabla \rightarrow -jk_0\mathbf{a}_k = -j\mathbf{k}_0 \quad (3-8)$$

if and only if all the dependent variables in Maxwell's equations, namely,  $\mathbf{H}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ , and  $\mathbf{E}$ , vary according to  $\exp [j(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{R})]$  and have constant amplitudes. Using Eq. (3-8) in Eqs. (1-3) and (1-4), and assuming  $\mathbf{B} = \mu_0\mathbf{H}$ , we obtain

$$\mathbf{H} = \frac{k_0\mathbf{a}_k \times \mathbf{E}}{\omega_0\mu_0} \quad (3-9)$$

and

$$\mathbf{D} = -\frac{k_0\mathbf{a}_k \times \mathbf{H}}{\omega_0} \quad (3-10)$$

where  $\mathbf{a}_k = \mathbf{k}_0/|\mathbf{k}_0| = \mathbf{k}_0/k_0$ . Finally, eliminating  $\mathbf{H}$  between Eqs. (3-9) and (3-10), we obtain

$$\mathbf{D} = \frac{k_0^2}{\omega_0^2\mu_0} [\mathbf{E} - (\mathbf{a}_k \cdot \mathbf{E})\mathbf{a}_k], \quad (3-11)$$

where we have used the vector identity

$$\mathbf{A} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{C} - (\mathbf{A} \cdot \mathbf{C})\mathbf{A}. \quad (3-12)$$

For electrically isotropic media, we know that

$$\mathbf{D} = \frac{k_0^2}{\omega_0\mu_0} \mathbf{E} = \frac{k_0^2}{\omega_0\mu_0} \mathbf{E} = \epsilon\mathbf{E}, \quad (3-13)$$

and hence  $\mathbf{a}_k \cdot \mathbf{E} = 0$ ; that is, there is no component of the electric field along the direction of propagation. However, in anisotropic crystals,  $\mathbf{D} = \epsilon\mathbf{E}$ ; hence Eq. (3-11) becomes

$$\epsilon\mathbf{E} = \frac{k_0^2}{\omega_0^2\mu_0} [\mathbf{E} - (\mathbf{a}_k \cdot \mathbf{E})\mathbf{a}_k] \quad (3-14)$$

Incidentally, from the first of Maxwell's equations [Eq. (1-1)], it follows, using Eq. (3-8), that  $\mathbf{a}_k \cdot \mathbf{D} = 0$ . This means that for a general anisotropic medium,  $\mathbf{D}$  is perpendicular to the direction of propagation, although

$\mathbf{E}$  may *not* be so because of the anisotropy. This is summarized in Fig. 3. Note that the Poynting vector  $\mathbf{S} = \mathbf{E} \times \mathbf{H}$  (which determines the direction of energy flow) is different from the direction of propagation of the wavefronts denoted by  $\mathbf{k}_0$ .

### 3.3 Applications of Birefringence: Wave Plates

Consider a plate made of a uniaxial material. The optic axis is along the  $z$  direction, as shown in Fig. 4. Let a linearly polarized real optical field incident on the crystal at  $x=0$  cause a field in the crystal at  $x=0^+$  of the form

$$\mathbf{E}_{\text{inc}} = \text{Re} [E_0 (\mathbf{a}_y + \mathbf{a}_z) e^{j\omega_0 t}]. \quad (3-15)$$

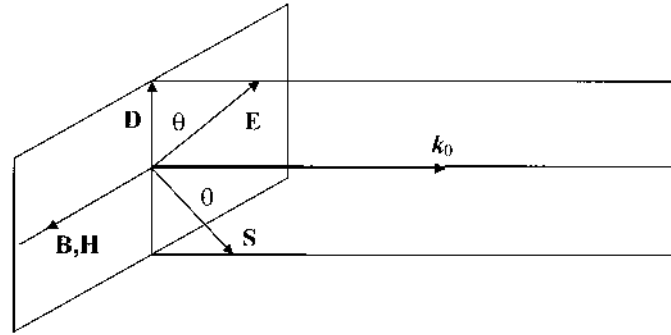
The wave travels through the plate, and at the right edge of the plate ( $x = L$ ) the real field can be represented as

$$\mathbf{E}_{\text{out}} = \text{Re} \left[ E_0 \left( \exp \left( -j \frac{\omega_0}{v_2} L \right) \mathbf{a}_y + \exp \left( -j \frac{\omega_0}{v_3} L \right) \mathbf{a}_z \right) \exp(j\omega_0 t) \right] \quad (3-16)$$

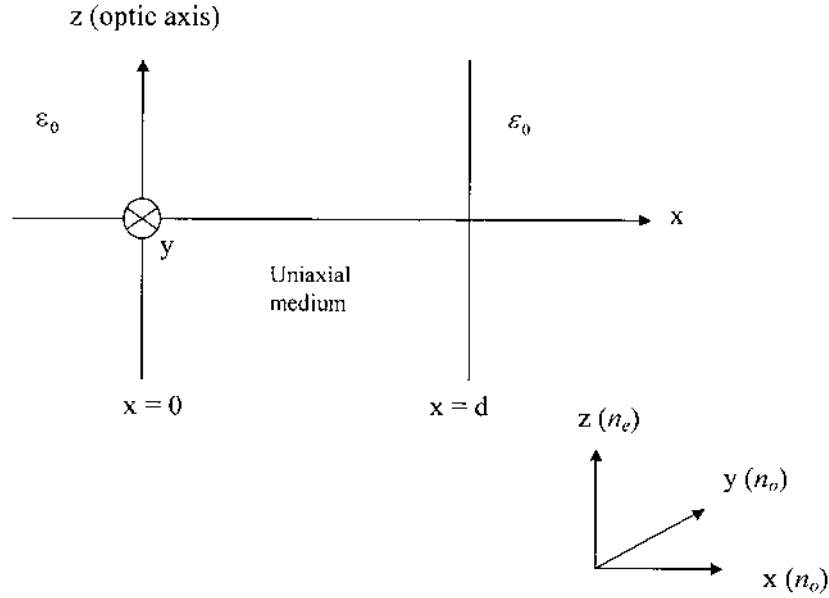
where  $v_2 = v_1 = 1/\sqrt{\mu_0 \epsilon_x} = 1/\sqrt{\mu_0 \epsilon_y}$  and  $v_3 = 1/\sqrt{\mu_0 \epsilon_z}$ .

Note that the two plane-polarized waves acquire a different phase as they propagate through the crystal. The relative phase shift  $\Delta\phi$  between the extraordinary and ordinary wave is

$$\Delta\phi = \left( \frac{\omega_0}{v_3} - \frac{\omega_0}{v_2} \right) L = \frac{\omega_0}{c} (n_e - n_o) L \quad (3-17)$$



**Figure 3** Diagram illustrating the direction of various field quantities in anisotropic crystals ( $\mathbf{S} = \mathbf{E} \times \mathbf{H}$  is the Poynting vector).



**Figure 4** A wave plate (retardation plate).

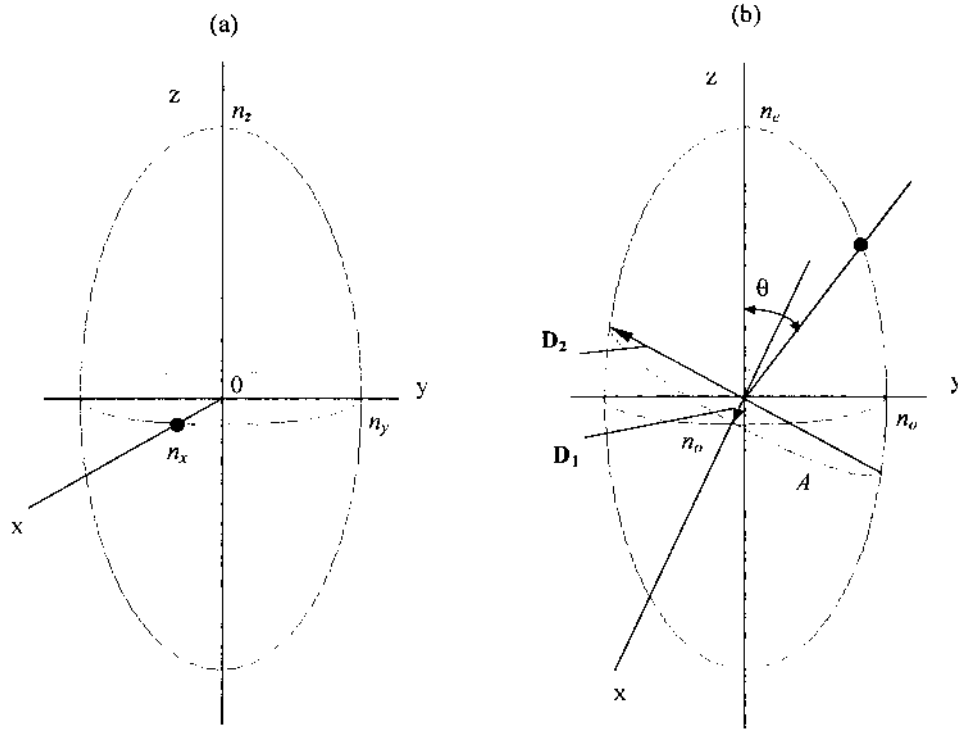
where  $c$  is the velocity of light in vacuum.  $n_o$ ,  $n_e$  are called the *ordinary* and *extraordinary* refractive indices, respectively. If  $n_e > n_o$ , the extraordinary wave lags the ordinary wave in phase; that is, the ordinary wave travels faster, whereas if  $n_e < n_o$ , the opposite is true. Such a phase shifter is often referred to as a *compensator* or a *retardation plate*. The directions of polarization for the two allowed waves are mutually orthogonal and are usually called the slow and fast axes of the crystal.

### 3.4 The Index Ellipsoid

For a plane-polarized wave propagating in any given direction in a uniaxial crystal, there are two allowed polarizations, one along the optic axis and the other perpendicular to it. The total phase velocity of a wave propagating in an arbitrary direction depends on the velocities of waves polarized solely along the directions of the principal axes and on the direction of propagation of the wave. A convenient method to find the directions of polarization of the two allowed waves and their phase velocities is through the *index ellipsoid*, a mathematical entity written as

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1, \quad (3-18)$$

where  $n_x^2 = \epsilon_x/\epsilon_0$ ,  $n_y^2 = \epsilon_y/\epsilon_0$ , and  $n_z^2 = \epsilon_z/\epsilon_0$ . Fig. 5(a) shows the index ellipsoid. From Eq. (3-18), we can determine the respective refractive indices as well as the two allowed polarizations of  $\mathbf{D}$  for a given direction of propagation in crystals. To see how this can be performed, consider a plane perpendicular to the direction of the propagation  $\mathbf{k}_0$ , containing the center of the ellipsoid. The intersection of the plane with the ellipsoid is the ellipse  $A$  in Fig. 5(b), drawn for a uniaxial crystal. The directions of the two possible displacements  $\mathbf{D}_1$  and  $\mathbf{D}_2$  now coincide with the major and minor axes of the ellipse  $A$ , and the appropriate refractive indices for the two allowed plane-polarized waves are given by the lengths of the two semiaxes. For uniaxial crystals  $\epsilon_x = \epsilon_y$ , and  $\epsilon_z$  is distinct. We then have  $n_x = n_y \equiv n_o$  and  $n_z \equiv n_e$ .  $\mathbf{D}_1$  is the ordinary wave and  $\mathbf{D}_2$  is the extraordinary wave. The index



**Figure 5** (a) Index ellipsoid and (b) two allowed polarizations  $\mathbf{D}_1$  and  $\mathbf{D}_2$ .

of refraction  $n_e(\theta)$  along  $\mathbf{D}_2$  can be determined using Fig. 6. Employing the relations

$$n_e^2(\theta) = z^2 + y^2 \quad (3-19a)$$

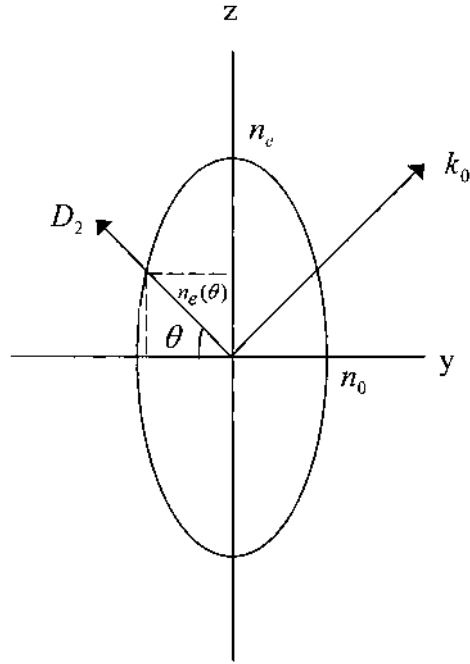
$$\frac{z}{n_e(\theta)} = \sin\theta, \quad (3-19b)$$

and the equation of the ellipse

$$\frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} = 1, \quad (3-19c)$$

we have

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_0^2} + \frac{\sin^2\theta}{n_e^2}. \quad (3-20)$$



**Figure 6** Cross-section of index ellipsoid illustrating the value of the refractive index depending on the direction of wave propagation.

From Fig. 6, we can immediately observe that when  $\theta = 0$ , i.e., the wave is propagating along the optic axis, no birefringence is observed [ $n_e(0) - n_0 = 0$ ]. Also, the amount of birefringence,  $n_e(\theta) - n_0$  depends on the propagation direction and it is maximum when the propagation direction is perpendicular to the optic axis,  $\theta = 90^\circ$ .

### 3.5 Electro-Optic Effect in Uniaxial Crystals

Having been introduced to wave propagation in anisotropic media, we are now in a position to analyze the *electro-optic effect*, which is inherently anisotropic. As seen below, we can effectively study this using the index ellipsoid concepts. In the following section, we will study the applications of the electro-optics, e.g., amplitude and phase modulation.

The *electro-optic effect* is, loosely speaking, the change in  $n_e$  and  $n_0$  that is linearly proportional to the applied field. Note that the *Pockels effect* can only exist in some crystals, namely, those that do not possess inversion symmetry [see Yariv, 1997]. The other case, namely, where  $n_e$  and  $n_0$  depend nonlinearly on the applied field, is called the *Kerr effect* and will be discussed in the following chapter.

Mathematically, the electro-optic effect can be best represented as a deformation of the index ellipsoid because of an external electric field. Thus Eq. (3-18), with  $n_x = n_y = n_0$  and  $n_z = n_e$ , represents the ellipsoid for uniaxial crystals in the absence of an applied field, i.e.,

$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} = 1 \quad (3-21)$$

where the directions  $x$ ,  $y$ , and  $z$  are the principal axes.

Restricting our analysis to the linear electro-optic (Pockels) effects, the general expression for the deformed ellipsoid is

$$\begin{aligned} & \left[ \frac{1}{n_0^2} + \Delta\left(\frac{1}{n^2}\right)_1 \right] x^2 + \left[ \frac{1}{n_0^2} + \Delta\left(\frac{1}{n^2}\right)_2 \right] y^2 + \left[ \frac{1}{n_e^2} + \Delta\left(\frac{1}{n^2}\right)_3 \right] z^2 \\ & + 2\Delta\left(\frac{1}{n^2}\right)_4 yz + 2\Delta\left(\frac{1}{n^2}\right)_5 xz + 2\Delta\left(\frac{1}{n^2}\right)_6 xy = 1 \end{aligned} \quad (3-22)$$

where

$$\Delta\left(\frac{1}{n^2}\right)_i = \sum_{j=1}^3 r_{ij} E_j, \quad i = 1, \dots, 6. \quad (3-23)$$

and where  $r_{ij}$ 's are called *linear electro-optic* (or *Pockels*) *coefficients*. The  $E_j$ 's are the components of the externally applied electric field in the  $x$ ,  $y$ , and  $z$

directions (to be distinguished from the optical fields represented as  $E_j$ 's). We can express Eq. (3-23) in matrix form as

$$\begin{bmatrix} \Delta\left(\frac{1}{n^2}\right)_1 \\ \Delta\left(\frac{1}{n^2}\right)_2 \\ \Delta\left(\frac{1}{n^2}\right)_3 \\ \Delta\left(\frac{1}{n^2}\right)_4 \\ \Delta\left(\frac{1}{n^2}\right)_5 \\ \Delta\left(\frac{1}{n^2}\right)_6 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad (3-24)$$

In Eqs. (3-22)–Eqs. (3-24), we have tacitly gone to the other convention (viz., 1, 2, 3 instead of  $x, y, z$ ) to comply with standard nomenclature. Note that when the applied field is zero, Eq. (3-22) reduces to Eq. (3-21). Equation (3-24) contains 18 elements and they are necessary in the most general case, when no symmetry is present in the crystal. Otherwise, some of the elements have the same value. Table 2 lists the nonzero elements of the linear electro-optic coefficients of some commonly used crystals. Using Eqs. (3-22) and

**Table 2** Electro-Optic Coefficients of Some Common Crystals

| Material                             | $r_{ij}$ ( $10^{-12}$ m/V)         | $\lambda_v$ ( $\mu\text{m}$ ) | Refractive index              |
|--------------------------------------|------------------------------------|-------------------------------|-------------------------------|
| LiNbO <sub>3</sub>                   | $r_{13} = r_{23} = 8.6$            | 0.63                          | $n_0 = 2.286$                 |
|                                      | $r_{33} = 30.8$                    |                               | $n_e = 2.200$                 |
|                                      | $r_{22} = -r_{61} = -r_{12} = 3.4$ |                               |                               |
|                                      | $r_{52} = r_{42} = 28$             |                               |                               |
| SiO <sub>2</sub>                     | $r_{11} = -r_{21} - r_{62} = 0.29$ | 0.63                          | $n_0 = 1.546$                 |
|                                      | $r_{41} = -r_{52} = 0.2$           |                               | $n_e = 1.555$                 |
| KDP (Potassium dihydrogen phosphate) | $r_{41} = r_{52} = 8.6$            | 0.55                          | $n_0 = 1.51$                  |
|                                      | $r_{63} = 10.6$                    |                               | $n_e = 1.47$                  |
| ADP (Ammonium dihydrogen phosphate)  | $r_{41} = r_{52} = 2.8$            | 0.55                          | $n_0 = 1.52$                  |
|                                      | $r_{63} = 8.5$                     | 0.55                          | $n_e = 1.48$                  |
| GaAs                                 | $r_{41} = r_{52} = r_{63} = 1.2$   | 0.9                           | $n_0 = n_e = 3.42$<br>(Cubic) |

(3-24) and Table 2, we can find the equation of the index ellipsoid in the presence of an external applied field. For instance, for a KDP crystal in the presence of an external field  $\mathbf{E} = E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z$ , the index ellipsoid equation can be reduced to

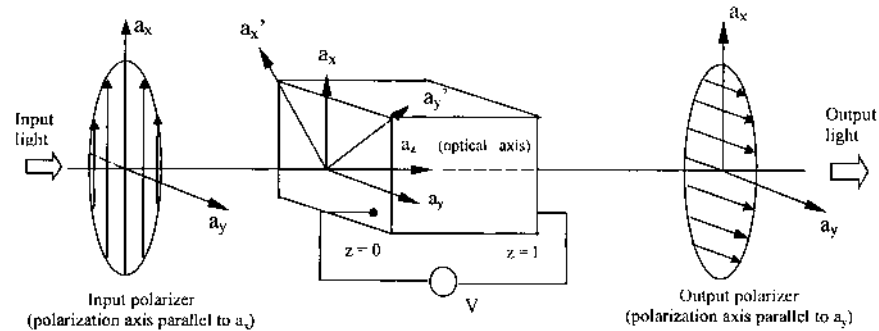
$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} + 2r_{41}E_x yz + 2r_{41}E_y xz + 2r_{63}E_z xy = 1. \quad (3-25)$$

The mixed terms in the equation of the index ellipsoid imply that the major and minor axes of the ellipsoid, with a field applied, are no longer parallel to the  $x$ ,  $y$ , and  $z$  axes, which are the directions of the principal axes when no field is present. This deformation of the index ellipsoid creates the externally induced birefringence.

### 3.6 Amplitude Modulation

The optical field can be modulated using an electro-optic modulator by applying the modulating electrical signal across the crystal to impose an external electric field. This changes the crystal properties through the linear electro-optic effect. There are two commonly used configurations, *longitudinal*, where the voltage is applied along the direction of optical propagation, and *transverse*, where it is applied transverse to optical propagation.

In what follows, we exemplify the operation of an electro-optic modulator by discussing an *amplitude modulator* in the longitudinal configuration. This is shown in Fig. 7. It consists of an electro-optic crystal placed between two crossed polarizers whose polarization axes are perpendicular to each other. The *polarization axis* defines the direction along which



**Figure 7** A longitudinal electrooptic intensity modulation system.

emerging light is linearly polarized. We use a KDP crystal with its principal axes aligned with  $x$ ,  $y$ , and  $z$ . An electric field is applied through the voltage  $V$  along the  $z$  axis, which is the direction of propagation of the optical field, thus justifying the name *longitudinal configuration*. Equation (3-25) then becomes

$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} + 2r_{63}E_zxy = 1. \quad (3-26)$$

By inspection, the ellipsoid has principal axes along the coordinates  $x'$ ,  $y'$ , and  $z$ , where

$$x' = \frac{1}{\sqrt{2}}(x - y), \quad y' = \frac{1}{\sqrt{2}}(x + y). \quad (3-27)$$

Introducing these expressions into Eq. (3-26), we have the index ellipsoid equation in the coordinate system aligned with the new principal axes:

$$\left(\frac{1}{n_x}\right)^2 x'^2 + \left(\frac{1}{n_y}\right)^2 y'^2 + \frac{z^2}{n_e^2} = 1. \quad (3-28)$$

In Eq. (3-28),

$$\frac{1}{n_x'^2} = \frac{1}{n_0^2} - r_{63}E_z, \quad (3-29)$$

implying

$$n_x' \approx n_0 + \frac{n_0^3}{2}r_{63}E_z. \quad (3-30)$$

Similarly,

$$n_y' \approx n_0 + \frac{n_0^3}{2}r_{63}E_z. \quad (3-31)$$

Also,

$$n_z = n_e. \quad (3-32)$$

The input (optical) field  $\text{Re} \{E_0 \exp j\omega_0 t\} \mathbf{a}_x$  at  $z=0$  is polarized along the  $x$  direction and can be resolved into two mutually orthogonal components polarized along  $x'$  and  $y'$ . After passage through the electro-optic crystal, the field components are

$$E_{x'} \Big|_{z=L} = \text{Re} \left[ \frac{E_0}{\sqrt{2}} \exp \left( j \left[ \omega_0 t - \frac{\omega_0}{c} n_x' L \right] \right) \right] \quad (3-33a)$$

$$E_{y'} \Big|_{z=L} = \text{Re} \left[ \frac{E_0}{\sqrt{2}} \exp \left( j \left[ \omega_0 t - \frac{\omega_0}{c} n'_y L \right] \right) \right] \quad (3-33b)$$

The phase difference at  $z = L$  between the two components is called the *retardation*  $\phi_L$  and is given by

$$\phi_L = \frac{\omega_0}{c} (n'_x - n'_y) L = \frac{\omega_0}{c} n_0^3 r_{63} V = \frac{2\pi}{\lambda_v} n_0^3 r_{63} V, \quad (3-34)$$

where  $V = E_z L$ .

At this point, it is interesting to mention that at  $\phi_L = \pi$ , the electro-optic crystal essentially acts as a *half-wave plate* where the birefringence is electrically induced. The crystal causes a  $x$ -polarized wave at  $z = 0$  to acquire a  $y$  polarization at  $z = L$ . The input light field then passes through the output polarizer at  $z = L$ . The optical field passes through the output polarizer unattenuated. With the electric field inside the crystal turned off ( $V = 0$ ), there is no output light, as it is blocked off by the crossed output polarizer. Therefore the system can switch light on and off electro-optically. The voltage yielding a retardation  $\phi_L = \pi$  is often referred to as the *half-wave voltage*,

$$V_\pi = \frac{\lambda_v}{2\pi n_0^3 r_{63}}. \quad (3-35)$$

From Table 2 and at  $\lambda_1 = 0.55 \mu\text{m}$ ,  $V_\pi \approx 7.5\text{kV}$  for KDP.

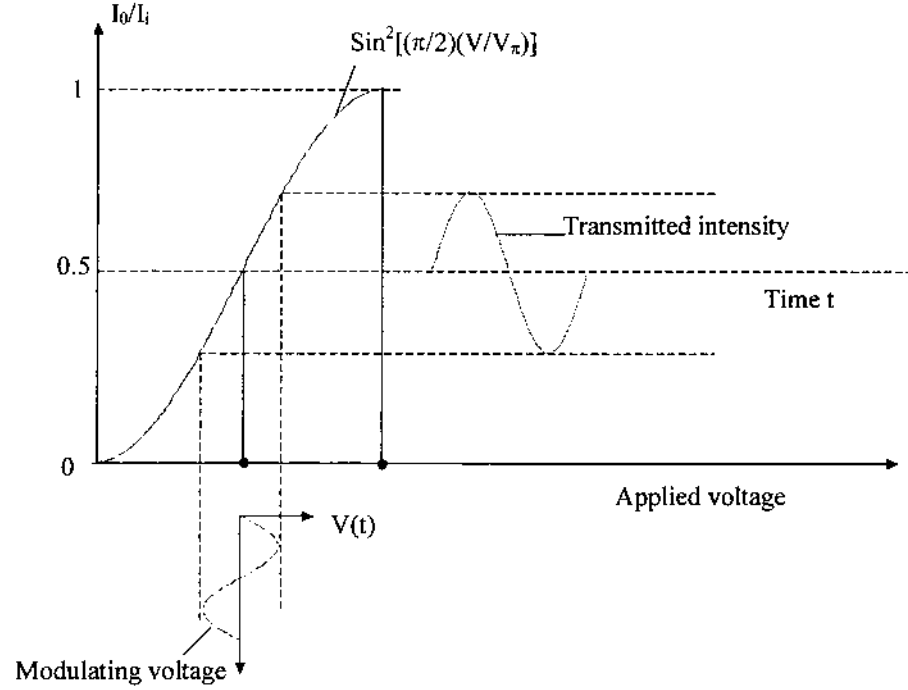
Returning to analyze the general system, the  $\mathbf{E}$  component parallel to  $\mathbf{a}_y$ , i.e., the component passed by the output polarizer, is

$$\begin{aligned} E_y \Big|_{z=L} &= \frac{E'_x - E'_y}{\sqrt{2}} \Big|_{z=L} = \frac{1}{2} \text{Re} \left\{ E_0 \left[ \exp \left( j \left[ \omega_0 t - \frac{\omega_0}{c} n'_x L \right] \right) \right. \right. \\ &\quad \left. \left. - \exp \left( j \left[ \omega_0 t - \frac{\omega_0}{c} n'_y L \right] \right) \right] \right\}. \end{aligned} \quad (3-36)$$

The ratio of the output ( $I_{\text{out}} \propto E_y^2$ ) and the input ( $I_{\text{in}} \propto E_0^2$ ) intensities is

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \sin^2 \left[ \frac{\omega_0}{2c} (n'_x - n'_y) L \right] = \sin^2 \left( \frac{\phi_L}{2} \right) = \sin^2 \left( \frac{\pi}{2} \frac{V}{V_\pi} \right) \quad (3-37)$$

Fig. 8 shows a plot of the transmission factor ( $I_0/I_i$ ) vs. the applied voltage. Note that the most linear region of the curve is obtained for a bias voltage at  $V_\pi/2$ . Therefore the electro-optic modulator is usually biased with a fixed retardation by the 50% transmission point. The bias can be achieved either electrically (by applying a fixed voltage  $V = \frac{1}{2} V_\pi$ ) or optically (by using a *quarter-wave plate*). The quarter-wave plate has to be inserted between the



**Figure 8** Relationship between the modulating voltage and the output intensity.

electro-optic crystal and the output polarizer in such a way that its slow and fast axes are aligned in the direction of  $\mathbf{a}'_x$  and  $\mathbf{a}'_y$ .

## 4 DIFFRACTION

So far in this chapter, we have only examined propagation of uniform plane waves. However, in practice, most optical waves are profiled and, consequently, may change their shape during propagation through an isotropic or anisotropic medium. In this section, we will review some of the fundamental concepts of diffraction, which causes the spreading of light beams.

### 4.1 The Spatial Transfer Function and Fresnel Diffraction

We start from the wave equation, Eq. (2-13). We assume that the total scalar field  $\psi(x, y, z, t)$  comprises a complex envelope  $\psi_e(x, y, z)$  riding on a

carrier of frequency  $\omega_0$  and propagation constant  $\mathbf{k}_0$  that propagates in the  $+z$  direction:

$$\psi(x, y, z; t) = \text{Re}\{\psi_e(x, y, z)\exp[j(\omega_0 t - k_0 z)]\}, \quad \frac{\omega_0}{k_0} = v. \quad (4-1)$$

Note that the complex envelope  $\psi_e$  is related to the phasor  $\psi_p$  as

$$\psi_p(x, y, z) = \psi_e(x, y, z)\exp(-jk_0 z), \quad (4-2)$$

and we will use one or the other according to convenience. Note also that

$$|\psi_e(x, y, z)| = |\psi_p(x, y, z)| = a(x, y, z), \quad (4-3)$$

Substituting Eq. (4.1) into Eq. (2-13) and assuming that  $\psi_e(x, y, z)$  is a slowly varying function of  $z$  (the direction of propagation) in the sense that

$$\frac{|\partial^2 \psi_e / \partial z^2|}{|\partial \psi_e / \partial z|} \ll k_0, \quad (4-4)$$

we obtain the *paraxial wave equation*,

$$2jk_0 \frac{\partial \psi_e}{\partial z} = \frac{\partial^2 \psi_e}{\partial z^2} + \frac{\partial^2 \psi_e}{\partial y^2}. \quad (4-5)$$

Equation (4-5) describes the propagation of the envelope  $\psi_e(x, y, z)$ , starting from an initial profile

$$\psi_e|_{z=0} = \psi_{e0}(x, y).$$

We now solve Eq. (4-5) using Fourier-transform techniques. As we will see shortly, this helps us understand diffraction from a simple transfer function concept of wave propagation, and also yields the Fresnel diffraction formula.

Assuming  $\psi_e$  to be Fourier-transformable, we employ the definition of the Fourier transform enunciated in Banerjee and Poon (1991) and its properties to Fourier-transform Eq. (4-5). This gives

$$\frac{d\Psi_e}{dz} = \frac{1}{2k_0} (k_x^2 + k_y^2) \Psi_e, \quad (4-6)$$

where  $\Psi_e(k_x, k_y; z)$  is the Fourier transform of  $\psi_e(x, y, z)$ ; and  $k_x, k_y$  represent spatial frequency variables corresponding to  $x, y$ , respectively. We now readily solve Eq. (4-6) to obtain

$$\Psi_e(k_x, k_y; z) = \Psi_{e0}(k_x, k_y) \exp\left(\frac{j(k_x^2 + k_y^2)z}{2k_0}\right), \quad (4-7)$$

where

$$\Psi_{e0}(k_x, k_y) = \Psi_e(k_x, k_y; z) = F_{xy}\{\psi_{e0}(x, y)\}, \quad (4-8)$$

We remark that  $\Psi_p(k_x, k_y; z) = \Psi_e(k_x, k_y; z)e^{-jk_0 z}$  is sometimes referred to as the angular plane-wave spectrum of  $\psi_e(x, y, z)$  [Goodman (1996)]. We can interpret Eq. (4-7) in the following way: Consider a linear system with  $\Psi_{e0}(k_x, k_y)$  as its input spectrum (i.e., at  $z = 0$ ) and where the output spectrum is  $\Psi_e(k_x, k_y; z)$ . Then, the spatial frequency response of the linear system is given by

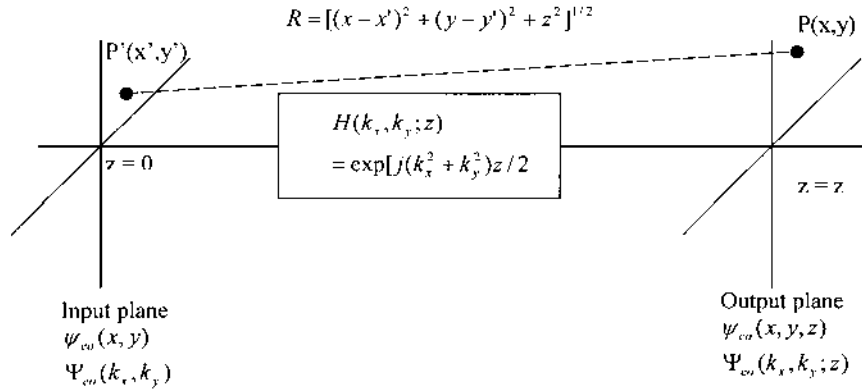
$$\frac{\Psi_e}{\Psi_{e0}} \triangleq H(k_x, k_y; z) = \exp\left[\frac{j(k_x^2 + k_y^2)z}{2k_0}\right]. \quad (4-9)$$

We will call  $H(k_x, k_y; z)$  the spatial transfer function of propagation of light through a distance  $z$  in the medium as shown in Fig. 9. The spatial impulse response is given by

$$h(x, y, z) = F_{xy}^{-1} H(k_x, k_y; z) = \frac{jk_0}{2\pi z} \exp\left[-\frac{j(x^2 + y^2)k_0}{2z}\right]. \quad (4-10)$$

Thus by taking the inverse Fourier transform of Eq. (4-7) and using Eq. (4-10), we obtain

$$\begin{aligned} \psi_e(x, y, z) &= \psi_{e0}(x, y) * h(x, y, z) \\ &= \frac{jk_0}{2\pi z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{e0}(x', y') \\ &\quad \times \exp\left[-j\frac{k_0}{2z} \left\{ (x - x')^2 + (y - y')^2 \right\}\right] dx' dy', \end{aligned} \quad (4-11)$$



**Figure 9** Block diagrammatic representation of the spatial transfer function of propagation. The input and output planes have primed and unprimed coordinates, respectively.

where  $*$  denotes convolution. The way Eq. (4-11) has been written indicates that the transverse coordinates are  $x', y'$  in the input plane and  $x, y$  in the output plane at the distance  $z$  away from the input plane (see Fig. 9). Using Eq. (4-2), we can write, from Eq. (4-11),

$$\begin{aligned} \psi_p(x, y, z) &= \frac{jk_0}{2\pi z} \exp[-jk_0 z] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{p0}(x', y') \\ &\quad \times \exp\left[-j\frac{k_0}{2z} \left\{ (x-x')^2 + (y-y')^2 \right\}\right] dx' dy', \end{aligned} \quad (4-12)$$

where

$$\psi_{p0}(x, y) = \psi_p(x, y, z)|_{z=0} = \psi_e(x, y, z)|_{z=0} = \psi_{e0}(x, y).$$

Equation (4-11) or (4-12) is termed the *Fresnel diffraction formula* and describes the *Fresnel diffraction* of a beam during propagation and having an arbitrary initial complex amplitude profile  $\psi_{e0}$ .

## 4.2 Diffraction of a Gaussian Beam

A *Gaussian beam* is one of the most important optical profiles because it is the spatial profile of the lowest-order mode of a laser. We will now study the behavior of Gaussian beams during propagation in a linear, isotropic, homogeneous medium. Extension to nonlinear media will be performed later.

We denote the Gaussian beam at  $z = 0$  in the form

$$\psi_{e0}(x, y) = \exp\left[-\frac{(x^2 + y^2)}{w_0^2}\right]. \quad (4-13)$$

Consequently,

$$\Psi_{e0}(k_x, k_y) = \pi w_0^2 \exp\left[-\frac{(k_x^2 + k_y^2) w_0^2}{4}\right], \quad (4-14)$$

and hence

$$\begin{aligned} \Psi_e(k_x, k_y; z) &= \Psi_{e0}(k_x, k_y) = H(k_x, k_y; z) \\ &= \pi w_0^2 \exp\left[-\frac{(k_x^2 + k_y^2) w_0^2}{4}\right] \exp\left[\frac{j(k_x^2 + k_y^2) z}{2k_0}\right] \\ &= \pi w_0^2 \exp\left[\frac{j(k_x^2 + k_y^2) z}{2k_0}\right], \end{aligned} \quad (4-15)$$

where

$$q \triangleq z + \frac{jk_0 w_0^2}{2} \triangleq z + jz_R \quad (4-16)$$

is called the  $q$  parameter of the Gaussian beam. The propagation of a Gaussian beam is completely described by the transformation of its  $q$  parameter. For instance, it is easy to see from Eqs. (4-15) and (4-16) that if a Gaussian beam propagates through a distance  $L$  in free space, its  $q$  changes to  $q'$ , where

$$q' = q + L \quad (4-17)$$

Now, to find the evolution of the Gaussian beam during propagation, we need to find the inverse Fourier transform of Eq. (4-15). Note that this equation is of the same form as Eq. (4-9), with  $z$  replaced by  $q$ . Hence

$$\begin{aligned} \psi_e(x, y, z) &= \frac{jk_0 w_0^2}{2q} \exp\left[-\frac{jk_0 w_0^2}{2q}\right] \exp\left[-\frac{j(x^2 + y^2)k_0}{2q}\right] \\ &= \left(\frac{w_0}{w(z)}\right) \exp\left\{\frac{x^2 + y^2}{w^2(z)}\right\} \exp\left\{-\frac{jk_0}{2R}(x^2 + y^2)\right\} \exp(-j\phi), \end{aligned} \quad (4-18)$$

where

$$w^2(z) = \frac{2z_R}{k_0} \left[1 + \left(\frac{z}{z_R}\right)^2\right] \quad (4-19)$$

$$R(z) = \frac{z^2 + z_R^2}{z} \quad (4-20)$$

$$\phi(z) = -\tan^{-1}\left(\frac{z}{z_R}\right). \quad (4-21)$$

Inspection of Eqs. (4-18)–Eqs. (4-21) shows that the magnitude of the diffracted Gaussian profile is still Gaussian, albeit with a decrease in amplitude during propagation and an increase in its *width*  $w$  (initially equal to the *waist* size  $w_0$ ). Observe that at  $z = z_R = k_0 w_0^2/2$ ,  $w^2 = 2w_0^2$ ; this distance is called the *Rayleigh length*. The remaining terms indicate *phase curvature*, with the *radius of curvature*  $R$  depending on the distance of propagation and an additional phase shift  $\phi$ . Note that the radius of curvature is always positive and indicates diverging wavefronts. Furthermore, by differentiating Eq. (4-20) with respect to  $z$ , we can find the position of the minimum radius of curvatures to be at  $z = z_R$ . Thus starting at  $z=0$ , the radius of curvature first decreases from infinity (plane wavefronts) to a minimum value before starting to increase again. For large values of  $z$ , the radius of curvature is

approximately equal to  $z$ . Correspondingly, the amplitude for large  $z$  decreases according to  $1/z$ . These results are consistent with the fact that from a large distance, the Gaussian beam almost looks like a point source, radiating spherical waves.

### 4.3 Fraunhofer Approximation and Fourier Optics

Thus far, we have studied the effect of propagation on the amplitude and phase distributions of a Gaussian beam. In particular, we examined the *Fresnel diffraction pattern*, which is determinable through the Fresnel diffraction formula [Eq. (4-11) or (4-12)]. The range of applicability of this formula is from distances not too close to the source (typically from about 10 times the wavelength). However, it turns out that it is not always easy to determine the diffraction pattern. For instance, the Fresnel diffraction pattern of a rectangular aperture cannot be expressed in analytical form. In this section, we examine a means of calculating the diffraction pattern at distances far away from the source or aperture. More precisely, note that if our observation plane is in the *far field*, i.e.,

$$\frac{k_0(x'^2 + y'^2)_{\max}}{2} = z_R \ll z, \quad (4-22)$$

where  $z_R$  is the *Rayleigh range*, then the value of the exponential  $[\exp -jk_0(x'^2 + y'^2)_{\max}]/2z$  is approximately unity over the input plane  $(x', y')$ . Under this assumption, which is commonly called the *Fraunhofer approximation*, Eq. (4-12) becomes

$$\begin{aligned} \psi_p &= \frac{jk_0 e^{-jk_0 z}}{2\pi z} \exp \left[ -j \frac{k_0}{2z} x^2 + y^2 \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{e0}(x', y') \\ &\quad \times \exp \left[ j \frac{k_0}{z} (xx' + yy') \right] dx' dy' \\ &= \frac{jk_0 e^{-jk_0 z}}{2\pi z} \exp \left[ -j \frac{k_0}{2z} (x^2 + y^2) \right] F_{xy} \{ \psi_{e0}(x, y) \} \Big|_{\substack{k_x = k_0 x/z \\ k_y = k_0 y/z}}, \end{aligned} \quad (4-23)$$

where  $F_{xy}$  denotes the two-dimensional Fourier transform operator, defined in Poon and Banerjee (2001). Equation (4-23) is termed the *Fraunhofer diffraction formula* and is the limiting case of the Fresnel diffraction studied earlier. The first exponential in Eq. (4-23) is the result of the phase change due to propagation, whereas the second exponential indicates a phase curvature that is quadratic in nature. Note that if we are treating diffraction of red light ( $\lambda_0 = 633$  nm) and the maximum dimensions on the input plane are 1 mm, the  $z \gg 5$  m. As a matter of fact, Fraunhofer diffraction can be observed at distances much smaller than the value just predicted. Examples

of Fraunhofer diffraction patterns from various apertures on initial amplitude/phase can be found in Banerjee and Poon (1991), Goodman (1996), and Poon and Banerjee (2001).

## 5 DISPERSION

In the previous section, we have analyzed the spreading of a beam due to diffraction. In this section, we will discuss another consequence of propagation: the spreading of pulses due to *dispersion*. In fact, diffraction and dispersion have many commonalities, especially when it comes to analyzing the propagation of Gaussian beams and the propagation of Gaussian pulses, as will be seen later.

Thus far, we have studied the propagation of waves using the wave equation in a medium whose properties, described by, say the permittivity or the phase velocity, have been assumed to be a constant with respect to the temporal frequency of the wave. In many cases, however, the permittivity, or equivalently, the refractive index, can be a function of frequency or, equivalently, the wavelength of the wave. This dependence of  $n$  on  $\omega$ , or  $\omega$  on  $k$ , the propagation constant, which is called dispersion, may be an intrinsic property of the material or topological in nature. The intrinsic properties could be contributed to

- (a) *orientational polarization*, where the dipoles align themselves when an external electric field is applied;
- (b) *electronic polarization*, where the electron cloud is distorted with respect to the nucleus when an electric field is applied; and
- (c) *ionic or atomic polarization*, where the positive and negative ions undergo a relative shift with the application of an external electric field.

Sometimes the dispersion may be caused by the geometry in which the wave is propagating, as for instance in a waveguiding structure, such as a fiber.

In what follows, we will outline a method to analyze wave propagation in a medium for which the dependence of  $\omega$  on  $k$  is specified. We will see that using the dispersion relation  $\omega = \omega(k)$ , one can derive the partial differential equation for the wavefunction such as the optical field.

Consider the simplest type of dispersion relation

$$\omega = vk. \quad (5-1)$$

The corresponding PDE can be found by substitution of the operators

$$\omega \rightarrow -j\frac{\partial}{\partial t}, \quad k \rightarrow j\frac{\partial}{\partial z} \quad (5-2)$$

into the dispersion relation. The reason for this is that if one thinks of the simplest type of propagating wave, such as a plane wave of the form  $\psi = \exp j(\omega t - kz)$ , then differentiation w.r.t.  $t$  gives  $j\omega\psi$ , while differentiation w.r.t.  $z$  gives  $-jk\psi$ . Hence upon applying the operators in Eq. (5-2) to the dispersion relation (5-1), we obtain the simplest one-dimensional *kinematic wave equation*

$$\partial\psi/\partial t + v\partial\psi/\partial z = 0. \quad (5-3)$$

Note that in this case,  $v = \omega/k = \partial\omega/\partial k$ . This means that the phase velocity  $\partial\omega/\partial k$  equals the group velocity  $\partial\omega/\partial k$ , and the medium is called *non-dispersive*. If  $\omega/k \neq \partial\omega/\partial k$ , the medium is *dispersive*. Extending the recipe to another dispersion relation

$$\omega = vk - \gamma k^3, \quad (5-4)$$

one obtains the evolution equation

$$\partial\psi/\partial t + v\partial\psi/\partial z + \gamma\partial^3\psi/\partial z^3 = 0 \quad (5-5)$$

which is the linear part of the *Korteweg–deVries equation*, a well known equation in hydrodynamics. Note that if the dispersion relation (5-1) is squared and the operator formalism used, the resulting PDE is the commonly seen *wave equation* in one spatial dimension [see Eq. (2-13)]:

$$\partial^2\psi/\partial t^2 - v^2\partial^2\psi/\partial z^2 = 0. \quad (5-6)$$

It can be shown that for the wavefunction  $\psi$  to be real, the dispersion relation  $\omega = \omega(k)$ , or equivalently,  $k = k(\omega)$ , must be an odd function of  $\omega$  and  $k$ , respectively. In the dispersion relation, a negative  $k$  does not necessarily mean that a wave propagates in one direction, while for positive  $k$  the wave propagates in the opposite direction. Rather, a wave propagating in the  $+z$  direction combines positive  $k$  with positive  $\omega$ , and negative  $k$  with negative  $\omega$ , provided that the phase velocity is taken to be positive. A wave propagating in the  $-z$  direction combines positive  $k$  with negative  $\omega$ , and vice versa. Now equations representing bidirectional propagation are usually of the quadratic type, as seen above for the wave equation, or the *waveguide dispersion* relation

$$\omega^2 = v^2k^2 + \omega_c^2 \quad (5-7)$$

where  $\omega_c$  represents the cutoff frequency of the waveguide. By carefully choosing the signs of  $\omega$  and  $k$ , this dispersion relation can represent wave propagation both along  $+z$  and  $-z$ . It can be readily verified that the dispersion relation  $\omega = \pm \text{sgn}(k)\sqrt{v^2k^2 + \omega_c^2}$ , where  $\text{sgn}(k)$  is the *signum function*, which is of the form  $\omega = \omega(k)$ , represents the correct dispersion relation for waves traveling along  $\pm z$  [Korpel and Banerjee (1984)].

So far we have concentrated on one-dimensional wave propagation. What happens if wave propagation is in higher dimensions? In the case when we can square the dispersion relation, as that performed to derive the wave equation [Eq. (5-6)] above, extension to higher dimensions can be simply performed by replacing the operator  $\partial^2/\partial z^2$  by the Laplacian  $\nabla^2$ . Thus the three-dimensional extension of the wave equation reads

$$\partial^2\psi/\partial t^2 - v^2\nabla^2\psi = 0 \quad (5-8)$$

as in Eq. (2-13). More difficult is the direct extension to higher dimensions without squaring the dispersion relation  $\omega = \omega(k)$  because  $\omega(k)$  may be a polynomial in odd powers of  $k$ . In this case, assuming weakly transverse paraxial behavior ( $k_x, k_y \ll k_z$ ), and nominally unidirectional propagation (along  $z$ ),

$$\begin{aligned} k &= \sqrt{k_x^2 + k_y^2 + k_z^2} = k_z + (1/2k_0)(k_x^2 + k_y^2) \\ &\rightarrow j\partial/\partial z + (j/2k_0)(\partial^2/\partial x^2 + \partial^2/\partial y^2) \end{aligned} \quad (5-9)$$

The PDE derived from Eq. (5-9) is of great importance because it describes the behavior of a pulsed beam in time and space. To see this more clearly, it can be shown that substitution of

$$\psi = \text{Re}[\psi_e(x, y, z, t)\exp(j\omega_0 t - jk_0 z)] \quad (5-10)$$

results in a PDE for the “slowly varying” envelope  $\psi_e$  of the form

$$2jk_0(v^{-1}\partial\psi_e/\partial t + \partial\psi_e/\partial z) = \partial^2\psi_e/\partial x^2 + \partial^2\psi_e/\partial y^2. \quad (5-11)$$

Equation (5-11) is well known. Without the term  $\partial\psi_e/\partial z$ , it is identical to the paraxial wave equation [Eq. (4-5)], and models the diffraction of beams in a medium.

Now let us look at a more complicated dispersion relation, one that is often used to describe pulse propagation in nonlinear optics (see Chap. 8). The relation

$$\omega = \omega(k) = \omega_0 + u_0(k - k_0) + \frac{u'_0}{2}(k - k_0)^2 + \dots, \omega > 0 \quad (5-12)$$

is often used to describe a small region on the general dispersion curve around the carrier frequency  $\omega_0$ . Here  $u_0$  is the group velocity and  $u'_0$  is called the group velocity dispersion. [Alternatively, the Taylor expansion  $k = k(\omega)$  is also often used, as will be seen in Chap. 8.] The appropriate extension for

negative frequencies can be easily made, using the properties of dispersion relations. Now, setting

$$\Omega = \omega - \omega_0, \quad \kappa = k - k_0, \quad (5-13)$$

where  $\Omega, \kappa$  refer to the sidebands around  $\omega_0, k_0$ , we obtain the dispersion relation for the envelope  $\psi_e$  of the wavefunction  $\psi$  as

$$\Omega = u_0 \kappa + \frac{u'_0}{2} \kappa^2 + \dots \quad (5-14)$$

For the one-dimensional case, substitution of operators for  $\Omega, \kappa$  gives the linear part of the *nonlinear Schrodinger equation*

$$j(\partial \psi_e / \partial t + u_0 \partial \psi_e / \partial z) - (u'_0/2) \partial^2 \psi_e / \partial z^2 = 0. \quad (5-15)$$

In a traveling frame of reference  $Z = z - u_0 t, T = t$ , Eq. (5-15) can be rewritten as

$$j \partial \psi_e / \partial T - (u'_0/2) \partial^2 \psi_e / \partial Z^2 = 0 \quad (5-16)$$

In higher dimensions, and assuming paraxial behavior, it can be shown that Eq. (5-16) modifies to [see Problem 11]

$$j \partial \psi_e / \partial T - (u'_0/2) \partial^2 \psi_e / \partial Z^2 - u_0/2k_0 \nabla_\perp^2 \psi_e = 0, \quad (5-17)$$

where  $\nabla_\perp^2$  denotes the transverse Laplacian.

The nonlinear extension of Eq. (5-17) is an important equation to study the propagation of pulsed beams in space and time in a nonlinear dispersive medium. For a comprehensive discussion on dispersion relations and operators and the material covered in this section, the reader is referred to Korpel and Banerjee (1984).

## 6 PROBLEMS

1. For a time-harmonic uniform plane wave in a linear, isotropic, homogeneous medium, the  $\mathbf{E}$  and  $\mathbf{H}$  fields vary according to  $\exp(j\omega_0 t - j\mathbf{K}_0 \cdot \mathbf{R})$ . Find the equivalents of the operators  $\partial/\partial t$  and  $\nabla$ , and hence reduce Maxwell's equations into an algebraic set of vector equations.
2. A circularly polarized electromagnetic wave in space is described through its  $\mathbf{E}$  field as

$$\mathbf{E} = 5 \cos(\omega_0 t - k_0 z) \mathbf{a}_x + 5 \sin(\omega_0 t - k_0 z) \mathbf{a}_y.$$

The wavelength is  $6 \times 10^{-7}$  m. Find the corresponding magnetic field and find the time-averaged Poynting vector.

3. Find the general traveling wave time-harmonic solutions to the scalar wave equation [Eq. (2-13)] in cylindrical and spherical coordinate systems. Assume angular independence in each case for simplicity.
4. Use the electromagnetic boundary conditions, viz., the continuity of the tangential components of  $\mathbf{E}$  and  $\mathbf{H}$ , and the continuity of the normal components of  $\mathbf{D}$  and  $\mathbf{B}$  across an interface, to show that when an optical wave encounters an interface, there is no change in the temporal frequency of the reflected and transmitted waves.
5. A circularly polarized plane wave of wavelength  $0.633 \mu\text{m}$ , described by

$$\mathbf{E} = E_0 \cos(\omega_0 t - k_0 x) \mathbf{a}_y + E_0 \sin(\omega_0 t - k_0 x) \mathbf{a}_z$$

is incident on  $x$ -cut calcite ( $n_o = 1.658$ ;  $n_e = 1.486$ ) of thickness  $6 \mu\text{m}$ . Find the expression for the emerging field and its state of polarization.

6. For a uniaxial crystal, show how an incident circularly polarized optical wave becomes periodically linearly polarized. An appropriate thickness  $L$  of such a crystal is a quarter-wave plate. Express  $L$  in terms of  $n_o$ ,  $n_e$  and the free-space wavelength.
7. From Eq. (3-26), show that the principal axes of the index ellipsoid in the presence of an electric field along the  $z$  direction are at  $45^\circ$  to the  $x$  and  $y$  directions.
8. Find the Fresnel diffraction pattern of a sinusoidal grating described by

$$t(x) = 0.5(1 + m \cos Kx).$$

Show that during propagation, there is periodic imaging of the transparency in the Fresnel region. Also, find the far-field diffraction pattern.

9. Derive the law for q-transformation for a Gaussian beam incident on a lens of focal length  $f$ . The phase transformation introduced by the lens is  $-(x^2 + y^2)k_0/2f$ , where  $k_0$  is the propagation constant in vacuum.

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# 2

## The Origin and Modeling of Optical Nonlinearity

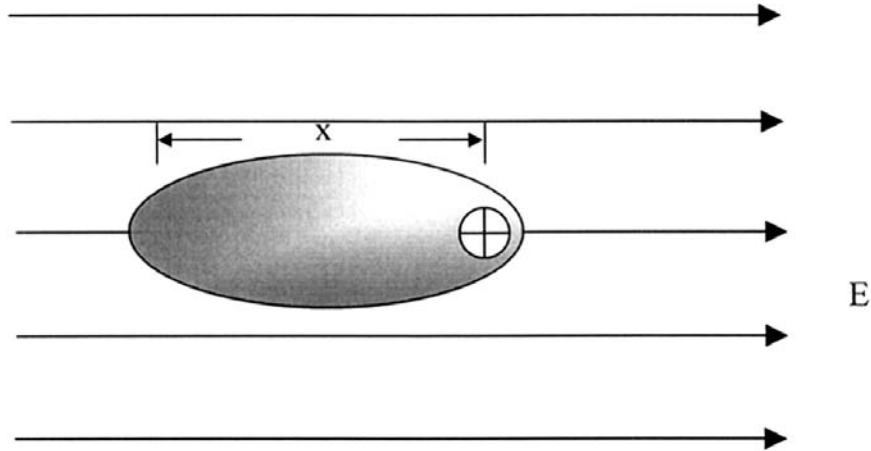
In Chap. 1, we reviewed the basic concepts of linear electromagnetic wave propagation in isotropic and anisotropic materials. However, optical propagation in many materials is nonlinear, in the sense that the polarization depends nonlinearly on the optical field in the medium. In order to understand the origin of such nonlinearities, we demonstrate, through a simple physical model of an electron around a nucleus, how the displacement of the electron, and hence the induced polarization, depends nonlinearly on the applied time-harmonic optical field. This explanation is, however, far too simplified to be used as a viable mathematical model in a practical nonlinear material. With rigor in mind, we introduce the time domain formulation of optical nonlinearity by relating the polarization to the optical field. A more widely used formulation is, however, found in the frequency domain. Thus, we use the Fourier transform to re-express the nonlinear polarizations in terms of the spectra of the optical fields and the so-called nonlinear susceptibilities. In most practical applications, the applied optical field is time-harmonic and so is the nonlinear polarization. We therefore provide the relations between the nonlinear polarization phasors and the optical field phasors. We also derive a nonlinear extension of the wave equation for the optical field in terms of the induced nonlinear polarization, assuming that both are expressible in terms of phasors. Finally, we illustrate another simple mathematical model for nonlinear propagation which has its foundations in fluid dynamics and compare with our more rigorously derived results.

## 1 A SIMPLE PHYSICAL MODEL FOR OPTICAL NONLINEARITY

Under the action of an applied electric field, the internal charge distribution within a dielectric is distorted. Specifically, in the case of electronic polarization, the field displaces the electron cloud with respect to the nucleus (see Fig.1) and hence creates a net dipole moment. In its simplest form, we can model the atom as a spring-mass system in which the displacement  $x$  of the electron from the nucleus can be compared to the extension of the spring. In the regime of *linear* optics, we assume the restoring mechanical force to be proportional to the displacement. If the applied electric field is time-harmonic, the differential equation satisfied by the displacement  $x(t)$  is

$$m_e \left[ \frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^{*2} x \right] = -eE_0 \cos \omega t \quad (1-1)$$

where  $e$  is the electronic charge and  $m_e$  is the electronic mass,  $E_0$  is the peak value of the applied electric field  $E(t)$  at the optical frequency  $\omega$ ,  $\omega^*$  represents the natural or resonant frequency of the oscillator, and  $\alpha$  is a damping constant. In the absence of the second term on the LHS, Eq. (1-1) is essentially the mathematical enunciation of Newton's second law relating the acceleration of the displaced particle to the net force (electrical and mechanical) on it. The second term models the damping of the oscillator, which in this case is due to the fact that atoms and molecules



**Figure 1** Distortion of an electron cloud in response to applied optical field.

in rather close proximity experience strong mutual interactions and consequently an appreciable frictional or damping force leading to the absorption of energy.

We can explicitly solve for  $x(t)$  in Eq. (1-1) by writing  $x(t) = \text{Re}[A \exp j\omega t]$  and then equating the coefficients of  $\exp j\omega t$ . In other words, we assume that the electric dipole oscillates at the same frequency as the incident optical field. This gives  $A = -(eE_0/m_e)/[\omega^{*2} + 2j\alpha\omega - \omega^2]$ . Hence

$$x(t) = -(eE_0/2m_e)[\exp j\omega t]/[\omega^{*2} + 2j\alpha\omega - \omega^2] + \text{c.c.} \quad (1-2)$$

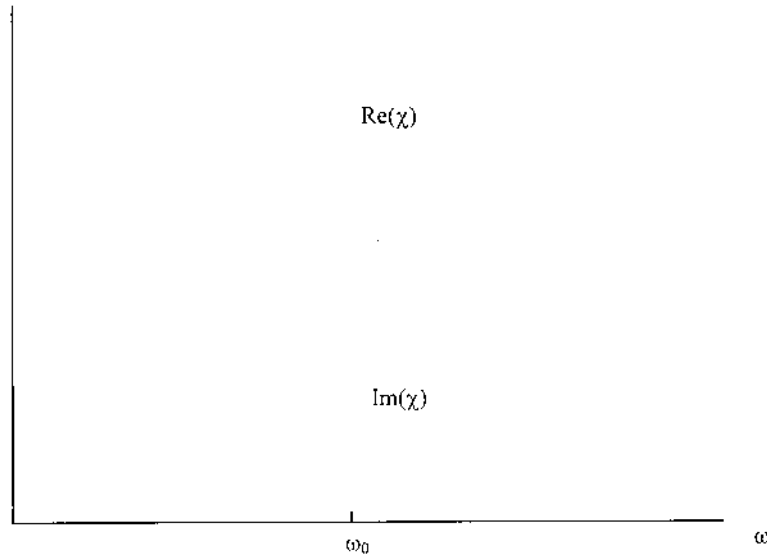
Now the dipole moment is equal to  $-ex(t)$ , and if there are  $N$  electrons per unit volume, we can write the electronic polarization  $P(t) = -eNx(t)$  in the form

$$P(t) = (1/2)\epsilon_0[\chi E_0 \exp j\omega t + \text{c.c.}] \quad (1-3)$$

where

$$\chi = \chi(\omega) = (Ne^2/\epsilon_0 m_e)/[\omega^{*2} + 2j\alpha\omega - \omega^2] \quad (1-4)$$

and  $\epsilon_0$  denotes the permittivity of free space. The real and imaginary parts of  $\chi$  are sketched in Fig. 2. Since the refractive index  $n(\omega) = [1 + \chi(\omega)]^{1/2}$ , it follows that the refractive index is dependent on frequency in a similar way. Since the



**Figure 2** Variation of the real and imaginary parts of the susceptibility with optical frequency.

real part of the refractive index varies with frequency, the material is, in general, dispersive, especially around the resonant frequency of the oscillator.

In the absence of absorption, i.e., when  $\alpha=0$ , we note from Eq. (1.3) that  $P = \epsilon_0 \chi E$  since  $\chi$  is purely real. The polarization is linearly dependent on the applied field. In the regime of *nonlinear* optics, this is no longer true. This is because the mechanical restoring force on the electron, call it  $F_m$  for brevity, cannot, in general, be assumed to be a linear function of the displacement of the electron. Linearity only holds if the displacement is small. For larger values of the displacement of  $x$ ,  $F_m$  may be significantly nonlinear with respect to  $x$ . The situation is analogous to that of an oscillating pendulum, which for small displacements also demonstrates simple harmonic motion. For larger values of the displacement, however,  $F_m$  is a sine function of the (angular) displacement. In the case of a crystal, the restoring force has additional nonlinear terms proportional to, for instance,  $x_2$  and  $x_3$ . In other words,  $F_m = -m_e \omega_0^2 x + Bx^2 + Cx^3 + \dots$ . However, for a symmetric crystal,  $B=0$  and so are the coefficients of all even powers of  $x$ . The reason for this is that the potential energy of an electron, which is the integral of  $F_m$ , is equal to  $-m_e \omega_0^2 x^2/2 + Bx^3/3 + Cx^4/4 + \dots$  in the general case. For a symmetric crystal,  $V(x) = V(-x)$  which implies that  $B=0$ . Now since the restoring force is, in general, nonlinear, the ODE in Eq. (1-1) describing the displacement of the electron is now modified to

$$m_e [d^2x/dt^2 + \omega^{*2}x] - Bx^2 - Cx^3 = -eE(t) = -eE_0 \cos \omega t \quad (1-5)$$

in the absence of absorption.

At this point, let us take a little time to discuss the nature of the solutions of Eq. (1-5) for a couple of special cases. Our results will help in formulating a general form for  $x(t)$ , and hence  $P(t)$ , in terms of  $E(t)$ . Consider first the case when  $B=0$  and  $E(t)=0$ , i.e., we are interested in the natural response of the nonlinear system. The solution procedure is rather tedious and involved and we will not present it here. We refer interested readers to Pipes and Harvill (1970). For initial conditions  $x(0)=a$  and  $dx(0)/dt=0$ , the solution is of the form

$$x(t) = a \cos \omega^{*'} t - (Ca^3/32m_e\omega^2)(\cos 3\omega^{*'} t - \cos \omega^{*'} t) + \dots \quad (1-6)$$

where

$$\omega^{*'}^2 = \omega^{*2} - 3a^2C/4m_e + \dots \quad (1-7)$$

Note that the natural response has a third harmonic term due to the presence of the cubic nonlinearity in the ODE, and, furthermore, the fundamental

frequency now depends on the amplitude  $a$  and is shifted from that in the linear case.

If, however, the above system, once again with  $B=0$ , has a sinusoidal driving term at frequency  $\omega$  [ $E(t) \neq 0$ ], the system is forced to respond at frequency  $\omega$  and its third harmonic  $3\omega$  due to the presence of the cubic nonlinearity. The resulting equation is known in mathematics as *Duffing's equation*. Experiments performed on dynamical systems which can be modeled by Duffing's equation show that as time  $t$  increases, the motion of the system indeed becomes periodic in the steady state after some initial transient motions have decayed. To solve Eq. (1-5) in this case, we follow the procedure detailed in Pipes and Harvill (1970). As a first approximation, we assume  $x = x_1 = a \cos \omega t$  in Eq. (1-5) and equate the coefficients of  $\cos \omega t$ . The resulting equation,

$$3Ca^3/4m_e\omega_0^2 = (1 - \omega^2/\omega^{*2})a - E_0/\omega^{*2}m_e \quad (1-8)$$

is a cubic equation in  $a$  which determines the amplitude of oscillation. We will not get into an involved discussion here on the dynamics of  $a$  as  $\omega$  is increased or decreased; it suffices to understand that for most values of  $\omega$ ,  $a$  has a unique value. If we now use this solution  $x_1$  to explicitly write down the nonlinear term in Eq. (1-5) as a second forcing term and solve the ODE for  $x$ , the second iterate for the solution, call it  $x_2$ , is

$$x_2(t) = a \cos \omega t - (Ca^3/36m_e\omega^2) \cos 3\omega t. \quad (1-9)$$

This second approximation can be resubstituted in the nonlinear part of Eq. (1-5) to derive the third iterate and so on. The process is convergent if  $C$  is sufficiently small.

The above example clearly shows that for forced oscillations at frequency  $\omega$ , the system response contains frequency components  $\omega$  and  $3\omega$ . In other words, the displacement  $x$ , and hence the polarization  $P$  of the cubically nonlinear system, will have contributions from  $E$  and  $E^3$ . In a similar way, one can show that Eq. (1-5) with  $C=0$  will have a solution containing frequencies  $\omega$  and  $2\omega$ , i.e., the response will have contributions from  $E$  and  $E^2$ . Hence, in general, we can expand the polarization  $P$  in the presence of nonlinearities in the form

$$P = \epsilon_0 [\chi^{(1)}E + \chi^{(2)}E^2 + \chi^{(3)}E^3] + \dots, \quad (1-10)$$

where  $\chi^{(1)}$  is the *linear susceptibility* and  $\chi^{(2)}$  and  $\chi^{(3)}$  are the *second- and third-order nonlinear susceptibilities*, respectively. We remind readers that for materials with inversion symmetry, all the even-order nonlinear susceptibility coefficients are equal to zero.

## 2 PHYSICAL EFFECTS OF NONLINEAR POLARIZATION

On the basis of the discussion in the previous section, we now point out some of the effects of the nonlinear dependence of the polarization on the optical field. In materials that do not possess inversion symmetry, the application of an optical field at frequency  $\omega$  will generate its second harmonic at  $2\omega$ . More effects arise when the applied field comprises two frequencies. For instance, if one of these fields is at frequency  $\omega$  while the other is essentially a d.c. bias, then the  $E^2$  term in  $P$  will cause a contribution to the polarization at frequency  $\omega$  whose amplitude depends on the strength of the d.c. bias. In other words, we can change the effective linear susceptibility  $\chi^{(1)}$ , and hence the refractive index, by applying a d.c. field across the material. This is the *linear electro-optic*, or *Pockels*, effect and is widely used in optical modulators. If the incident field comprises two frequencies,  $\omega_1$  and  $\omega_2$ , then the sum and difference frequencies  $\omega_1 \pm \omega_2$  are generated.

For materials with inversion symmetry, one of the effects of the nonlinear polarization is third harmonic generation. We can see this readily by examining the explicit solution to the displacement and hence the polarization in response to an external driving field [see Eq. (1-9)]. In addition, note that because of the cubic nonlinearity, the amplitude  $a$  of the displacement now nonlinearly depends on the amplitude of the external optical field. From Eq. (1-8), we can estimate  $a$  for small  $C$  as follows. The first approximation to  $a$  is  $E_0/\omega^2 m_e(1 - \omega^2/\omega^{*2})$ , derived by neglecting the nonlinear term in  $a$  in Eq. (1-8). If this is resubstituted on the LHS of Eq. (1-8), the second approximation to  $a$  reads

$$a = c_1 E_0 + c_3 E_0^3 \quad (2-1)$$

where

$$\begin{aligned} c_1 &= 1/\omega^2 m_e(1 - \omega^2/\omega^{*2}), \\ c_3 &= 3C/4\omega^8 m_e^4(1 - \omega^2/\omega^{*2})^2. \end{aligned} \quad (2-2)$$

Hence, around the frequency  $\omega$ , the induced polarization is equal to  $\epsilon_0 \chi_{\text{eff}}^{(1)} E$  where  $\chi_{\text{eff}}^{(1)}$ , the *effective* first order susceptibility, depends on the square of the amplitude  $E_0$  of the incident field. We can see this also from Eq. (1-10) with  $\chi^{(2)} = 0$  by setting  $E = E_0 \cos \omega t$ . It then readily follows that  $\epsilon_0 \chi_{\text{eff}}^{(1)} = \chi^{(1)} + (3/4)\chi^{(3)}E_0^2$ . We may interpret this differently by saying that the refractive index is now modified by an amount proportional to the magnitude squared, or the intensity, of the incident optical field. This is responsible for a wide variety of effects, viz., *self-focusing*, *self-bending*, and *self-phase modulation* of a laser beam (Chap. 4), *optical bistability* and *switching* (Chap. 5), *phase conjugation* (Chap. 6), and *soliton propagation* in nonlinear optical fibers (Chap. 8).

### 3 MATHEMATICAL MODELING OF OPTICAL NONLINEARITIES

Thus far in this chapter, we have provided a simple description of the physical origin of optical nonlinearities and listed a few of their physical effects. We have seen how the induced polarization becomes a nonlinear function of the applied optical field in an optically nonlinear medium. It is now time to examine the optical nonlinearities with a little more mathematical rigor. One of the reasons for this is to expose readers to the various mathematical formulations relating the polarization to the electric field that exist in the literature. Furthermore, most optically nonlinear materials are anisotropic and demand the use of tensors to describe their linear properties. Also, the linear and the nonlinear susceptibility tensors are, in general, functions of the interacting frequencies, so that one needs to exercise some caution while relating  $\mathbf{P}$  to  $\mathbf{E}$  in the time and frequency domains. Another objective of this section will be to decide on as simple a formulation as possible that will help us analyze the nonlinear optical effects stated above in the easiest possible way.

#### 3.1 Time Domain Formulation

To preserve mathematical rigor, we will reintroduce the polarization  $\mathbf{P}$  (now considered a vector) as

$$\mathbf{P}(t) = \mathbf{P}^{(1)}(t) + \mathbf{P}^{(2)}(t) + \mathbf{P}^{(3)}(t) + \dots \quad (3-1)$$

where  $\mathbf{P}^{(1)}$  is linear in the field and  $\mathbf{P}^{(2)}$  and  $\mathbf{P}^{(3)}$  are quadratically and cubically dependent on the field, respectively. In the time domain formulation, we will write

$$\mathbf{P}^{(1)}(t) = \epsilon_0 \int_{-\infty}^{\infty} d\tau \mathbf{X}^{(1)}(t; \tau) \mathbf{E}(\tau), \quad (3-2a)$$

or, using the Einstein convention, as

$$\mathbf{P}_i^{(1)}(t) = \epsilon_0 \int_{-\infty}^{\infty} d\tau X_{ij}^{(1)}(t; \tau) E_j(\tau), \quad (3-2b)$$

where  $\mathbf{X}^{(1)}$  is a second rank tensor which is a function of  $t$  and  $\tau$ . This equation is characteristic of a linear system whose response to an impulse  $\delta(t-\tau)$  is equal to  $\mathbf{X}^{(1)}(t; \tau)$ . We remind readers that if a linear system is time-invariant, then  $\mathbf{X}^{(1)}(t; \tau) = \mathbf{X}^{(1)}(t-\tau)$ . In that case, the integral in Eqs. (3-2a) and (3-2b) reduces to the convolution of  $\mathbf{X}^{(1)}$  and  $\mathbf{E}$ :

$$\mathbf{P}^{(1)}(t) = \epsilon_0 \int_{-\infty}^{\infty} d\tau \mathbf{X}^{(1)}(t-\tau) \mathbf{E}(\tau). \quad (3-3)$$

The Fourier transform of the response  $\mathbf{P}^{(1)}(t)$  is then the product of the Fourier transforms of  $\varepsilon_0 \mathbf{X}^{(1)}(t)$  and  $\mathbf{E}(t)$  and bears resemblance with the linear part of Eq. (1-10). More on this follows below in the discussion of the frequency domain formulation.

In the same way, we can express the second term,  $\mathbf{P}^{(2)}(t)$ , in Eq. (3-1) as

$$\mathbf{P}^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \mathbf{X}^{(2)}(t; \tau_1, \tau_2) : \mathbf{E}(\tau_1) \mathbf{E}(\tau_2) \quad (3-4a)$$

or, using the Einstein convention, as

$$\mathbf{P}_i^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 X_{ijk}^{(2)}(t; \tau_1, \tau_2) : E_j(\tau_1) E_k(\tau_2). \quad (3-4b)$$

Once again, if the system is time-invariant,  $\mathbf{X}^{(2)}(t; \tau_1, \tau_2) = \mathbf{X}^{(2)}(t - \tau_1, t - \tau_2)$  and Eqs. (3-4a) and (3-4b) reduce to

$$\mathbf{P}^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \mathbf{X}^{(2)}(t - \tau_1, t - \tau_2) : \mathbf{E}(\tau_1) \mathbf{E}(\tau_2). \quad (3-5)$$

Similarly, the third term  $\mathbf{P}^{(3)}(t)$ , for a time-invariant system, becomes

$$\begin{aligned} \mathbf{P}_i^{(3)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 d\tau_3 X_{ijkl}^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \\ : \mathbf{E}(\tau_1) \mathbf{E}(\tau_2) \mathbf{E}(\tau_3). \end{aligned} \quad (3-6)$$

Hence, we can express the total polarization  $\mathbf{P}$  in Eq. (3-1) in terms of the integrals in Eqs. (3-3), (3-5), (3-6) for a time-invariant system.

### 3.2 Frequency Domain Formulation

The time domain formulation of the nonlinear polarization can provide a complete description of the nonlinear optical properties of the medium. A more widely used description is, however, the frequency domain formulation which we describe below.

To begin, consider the linear part of the polarization  $\mathbf{P}^{(1)}(t)$  in Eq. (3-3). We can write this alternatively as

$$\mathbf{P}^{(1)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} d\tau \mathbf{X}^{(1)}(\tau) \mathbf{E}(t - \tau) \quad (3-7)$$

by a simple change of variable. If now we express  $\mathbf{E}(t)$  in terms of its Fourier transform  $\mathbf{E}(\omega)$  as

$$\mathbf{E}(t) = \int_{-\infty}^{\infty} d\omega \mathbf{E}(\omega) \exp j\omega t \quad (3-8)$$

we can rewrite Eq. (3-7) in the form

$$\begin{aligned}\mathbf{P}^{(1)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega \mathbf{E}(\omega) \mathbf{X}^{(1)}(\tau) \exp j\omega(t - \tau) \\ &= \varepsilon_0 \int_{-\infty}^{\infty} d\omega \chi^{(1)}(-\omega; \omega) \cdot \mathbf{E}(\omega) \exp j\omega t\end{aligned}\quad (3-9)$$

where

$$\chi^{(1)}(-\omega; \omega) = \int_{-\infty}^{\infty} d\tau \mathbf{X}^{(1)}(\tau) \cdot \exp -j\omega\tau. \quad (3-10)$$

In the above definition,  $\chi^{(1)}(-\omega; \omega)$  is the *linear susceptibility tensor* and is proportional to the Fourier transform of  $\mathbf{X}^{(1)}(t)$ . We have put in the additional argument  $-\omega$  in front of the semicolon in the definition of  $\chi^{(1)}$  to be consistent with the definition of higher-order nonlinear susceptibility tensors. Note that if we define  $\mathbf{P}^{(1)}(\omega)$  to be the Fourier transform of  $\mathbf{P}^{(1)}(t)$  in a manner identical to Eq. (3-8), then from Eq. (3-10), it follows that  $\mathbf{P}^{(1)}(\omega) = \varepsilon_0 \chi^{(1)}(-\omega; \omega) \mathbf{E}(\omega)$ , in agreement with our discussion following Eq. (3-3). If the optical field can be represented by means of a phasor, i.e.,  $\mathbf{E}(t) = (1/2) \mathbf{E}_p \exp j\omega_0 t + \text{c.c.}$ , then  $\mathbf{E}(\omega) = (1/2) [\mathbf{E}_p \delta(\omega - \omega_0) + \mathbf{E}_p^* \delta(\omega + \omega_0)]$  and  $\mathbf{P}^{(1)}(\omega) = (1/2) \varepsilon_0 \chi^{(1)}(-\omega; \omega) [\mathbf{E}_p \delta(\omega - \omega_0) + \mathbf{E}_p^* \delta(\omega + \omega_0)]$ . If we now also express  $\mathbf{P}^{(1)}(t)$  and take its Fourier transform, then it readily follows that  $\mathbf{P}_p^{(1)} = \varepsilon_0 \chi^{(1)}(-\omega_0; \omega_0) \mathbf{E}_p$ . In writing this relation, we have replaced  $\omega$  in the argument of  $\chi^{(1)}$  by  $\omega_0$  since in the expression for  $\mathbf{P}^{(1)}(\omega)$  above, the  $\chi^{(1)}$  is multiplied by a delta function. Finally, let us point out that even if the optical field phasor is a slowly varying function in the sense that  $|d\mathbf{E}_p(t)/dt| \ll \omega_0 |\mathbf{E}_p(t)|$ , one can rigorously show that the relation between  $\mathbf{P}_p^{(1)}(t)$  and  $\mathbf{E}_p(t)$  is  $\mathbf{P}_p^{(1)}(t) = \varepsilon_0 \chi^{(1)}(-\omega_0; \omega_0) \mathbf{E}_p(t)$ , which is similar to that derived above. Table 1 summarizes the differing definitions of the linear part of the polarization in time and frequency domain formulation, as well as the phasor form.

**Table 1** Time, Frequency, and Phasor Domain Formulations Showing the Dependence of the Linear Polarization on the Optical Field

| Regime                               | Formula                                                                                                         |
|--------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| Time domain formulation—general      | $\mathbf{P}^{(1)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} d\tau \mathbf{X}^{(1)}(\tau) \mathbf{E}(t - \tau)$ |
| Frequency domain formulation—general | $\mathbf{P}^{(1)}(\omega) = \varepsilon_0 \chi^{(1)}(-\omega; \omega) \mathbf{E}(\omega)$                       |
| Phasor form                          | $\mathbf{P}_p^{(1)}(t) = \varepsilon_0 \chi^{(1)}(-\omega_0; \omega_0) \mathbf{E}_p(t)$                         |

In a similar way, we can assess the behavior of the second-order polarization  $\mathbf{P}^{(2)}$  in the frequency domain. We first recast Eq. (3-5) in the form

$$\mathbf{P}^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \mathbf{X}^{(2)}(\tau_1, \tau_2) : \mathbf{E}(t - \tau_1) \mathbf{E}(t - \tau_2) \quad (3-11)$$

using a simple change of variables. Now using the definition

$$\begin{aligned} \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \mathbf{X}^{(2)}(\tau_1, \tau_2) \cdot \exp \\ &\quad -j(\omega_1 \tau_1 + \omega_2 \tau_2) \end{aligned} \quad (3-12)$$

for the quadratic susceptibility tensor, and by expressing the  $\mathbf{E}$ 's in Eq. (3-11) in terms of their Fourier transforms as defined through Eq. (3-8), we can re-express the above relation after straightforward algebra as

$$\begin{aligned} \mathbf{P}^{(2)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) \\ &\quad : \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \exp j(\omega_1 + \omega_2)t. \end{aligned} \quad (3-13)$$

The argument  $-(\omega_1 + \omega_2)$  in front of the semicolon in the definition of  $\chi^{(2)}$  is put conventionally to remind readers that the quadratic susceptibility in point is at a frequency which is the sum of the optical driving frequencies. Now, in order to derive the relation in the frequency domain, we express  $\mathbf{P}^{(2)}(\omega)$  in a manner analogous to Eq. (3-8) and recast Eq. (3-13) in the form

$$\begin{aligned} \mathbf{P}^{(2)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) \\ &\quad : \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \exp j\omega t \delta(\omega - \omega_1 - \omega_2). \end{aligned} \quad (3-14)$$

It then readily follows that

$$\begin{aligned} \mathbf{P}^{(2)}(\omega) &= \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) \\ &\quad : \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \delta(\omega - \omega_1 - \omega_2). \end{aligned} \quad (3-15)$$

In the linear case, it was straightforward to determine the relationship between the polarization phasor  $\mathbf{P}_p^{(1)}$  and the optical field phasor  $\mathbf{E}_p$ . Note that both the optical field and the linear polarization are at the same frequency  $\omega_0$ . This is, however, not true in the case of the second-order polarization since the product of the optical fields, as in Eqs. (3-11)–(3-15), will, in general, give rise to second-order polarizations at different frequencies. For instance, if the optical field is at a frequency  $\omega_0$ , the second-order polarization will have contributions at  $2\omega_0$  and 0. If the optical field comprises

two frequencies, e.g.,  $\omega_{01}$  and  $\omega_{02}$ , the resulting polarization will have contributions at  $\omega_{01} \pm \omega_{02}$ . To examine this in a little more detail, consider the case when the optical field comprises one frequency and can be written as  $\mathbf{E}(t) = (1/2)[\mathbf{E}_p^{\omega_0} \exp j\omega_0 t + \text{c.c.}]$ . If we now substitute this into Eq. (3-15), the RHS becomes

$$\begin{aligned} \text{RHS} &= \frac{\varepsilon_0}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) \\ &\quad : \left[ \mathbf{E}_p^{\omega_0} \delta(\omega_1 - \omega_0) + \mathbf{E}_p^{\omega_0*} \delta(\omega_1 + \omega_0) \right] \\ &\quad \times \left[ \mathbf{E}_p^{\omega_0} \delta(\omega_2 - \omega_0) + \mathbf{E}_p^{\omega_0*} \delta(\omega_2 + \omega_0) \right] \delta(\omega - \omega_1 - \omega_2). \end{aligned} \quad (3-16)$$

We can decompose this into four integrals arising from the product of the Fourier transforms of the optical field. The first integral, call it  $I_1$ , is of the form

$$\begin{aligned} I_1 &= \frac{\varepsilon_0}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) \\ &\quad : \mathbf{E}_p^{\omega_0} \delta(\omega_1 - \omega_0) \mathbf{E}_p^{\omega_0} \delta(\omega_2 - \omega_0) \delta(\omega - \omega_1 - \omega_2) \\ &= \frac{\varepsilon_0}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) \\ &\quad : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0} \delta(\omega_1 - \omega_0) \delta(\omega_2 - \omega_0) \delta(\omega - 2\omega_0). \end{aligned} \quad (3-17)$$

In writing the second line in Eq. (3-17) from the first, we have replaced  $\omega_1$  and  $\omega_2$ , in the arguments of  $\chi^{(2)}$  and of the third  $\delta$ -function, by  $\omega_0$  due to the presence of the first two  $\delta$ -functions. We can now directly integrate Eq. (3-17) and write

$$I_1 = \frac{\varepsilon_0}{4} \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0} \delta(\omega - 2\omega_0). \quad (3-18)$$

In a similar fashion, we can evaluate all the other integrals from Eq. (3-16). It is possible to combine  $I_{2,3}$  in the form

$$\begin{aligned} I_2 + I_3 &= \frac{\varepsilon_0}{4} \left[ \chi^{(2)}(0; -\omega_0, \omega_0) + \chi^{(2)}(0; \omega_0, -\omega_0) \right] \\ &\quad : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0*} \delta(\omega) \end{aligned} \quad (3-19)$$

and express  $I_4$  as

$$I_4 = \frac{\varepsilon_0}{4} \chi^{(2)}(2\omega_0; -\omega_0, -\omega_0) : \mathbf{E}_p^{\omega_0*} \mathbf{E}_p^{\omega_0*} \delta(\omega + 2\omega_0). \quad (3-20)$$

From Eqs. (3-18)–(3-20), it is clear that the second-order polarization has contributions at frequencies  $\pm 2\omega_0$  and 0. Thus, we write  $\mathbf{P}^{(2)}(t) = (1/2) [\mathbf{P}_p^{(2)\omega_0} \exp 2j\omega_0 t + \mathbf{P}_p^{(2)0} + \text{c.c.}]$  and find

$$\mathbf{P}_p^{(2)2\omega_0} = \frac{\varepsilon_0}{2} \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}, \quad (3-21)$$

$$\mathbf{P}_p^{(2)0} = \frac{\varepsilon_0}{4} [\chi^{(2)}(0; -\omega_0, \omega_0) + \chi^{(2)}(0; \omega_0, -\omega_0)] : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0*}. \quad (3-22)$$

There is an important property of the nonlinear susceptibility tensor which we will state here without proof. It can be shown that due to the intrinsic permutation symmetry of  $\mathbf{X}^{(2)}(\tau_1, \tau_2)$ ,  $\chi_{ijk}^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2) = \chi_{ikj}^{(2)}(-\omega_1 - \omega_2; \omega_2, \omega_1)$ . In other words,  $\chi^{(2)}$  is invariant under the interchange of the pairs  $(j, \omega_1)$  and  $(k, \omega_2)$ . For a detailed proof, we refer readers to Butcher and Cotler (1990). Now we can rewrite Eq. (3-22), using the Einstein convention, as

$$\mathbf{P}_{pi}^{(2)0} = \frac{\varepsilon_0}{4} [\chi_{ijk}^{(2)}(0; -\omega_0, \omega_0) : \mathbf{E}_{pj}^{\omega_0} \mathbf{E}_{pk}^{\omega_0*} + \chi_{ijk}^{(2)}(0; \omega_0, -\omega_0) : \mathbf{E}_{pj}^{\omega_0} \mathbf{E}_{pk}^{\omega_0*}].$$

If now we interchange the dummy variables  $j$  and  $k$  in the second term on the RHS and invoke the property of the  $\chi^{(2)}$  mentioned above, the second term becomes identical to the first term. This means that Eq. (3-22) may be simplified to

$$\mathbf{P}_p^{(2)0} = \frac{\varepsilon_0}{2} [\chi^{(2)}(0; -\omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0*}]. \quad (3-23)$$

A similar property holds for all higher-order susceptibilities, as will be mentioned later.

Note that in each of the cases discussed above, the superscript on  $\mathbf{P}^{(2)}$  matches with the argument of  $\chi^{(2)}$  before the semicolon, up to a sign term. Also, note the factor of  $(1/2)$  now multiplying the RHS of Eqs. (3-21) and (3-23). This factor depends on the interacting frequencies participating in the nonlinear interaction. For instance, in the case of sum and difference frequency mixings, e.g.,  $\omega_{01} \pm \omega_{02} \rightarrow \omega_{03}$ , where none of the participating frequencies are equal to zero, the value of this factor is 1. Thus, for sum frequency generation, the second-order polarization would have the form

$$\mathbf{P}_p^{(2)\omega_{03}} = \varepsilon_0 [\chi^{(2)}(-\omega_{03}; \omega_{01}, \omega_{02}) : \mathbf{E}_p^{\omega_{01}} \mathbf{E}_p^{\omega_{02}*}]. \quad (3-24)$$

In the case of the *Pockels effects*, also called the *linear electro-optic effect*, a factor of 2 should be put in front of the product between the requisite second-order susceptibility and the fields. Table 2 summarizes the definitions of the second-order polarization in the time and frequency domains and lists the relationships between some of the polarizations and the optical fields in

**Table 2** Time, Frequency, and Phasor Domain Formulations Showing the Dependence of the Second-Order Polarization on the Optical Field

| Regime                                  | Formula                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|-----------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Time domain<br>formulation—general      | $\mathbf{P}^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \mathbf{X}^{(2)}(\tau_1, \tau_2)$ $: \mathbf{E}(t - \tau_1) \mathbf{E}(t - \tau_2)$                                                                                                                                                                                                                                                                                |
| Frequency domain<br>formulation—general | $\mathbf{P}^{(2)}(\omega) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \chi^{(2)}(-\omega_1 - \omega_2; \omega_1, \omega_2)$ $: \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \delta(\omega - \omega_1 - \omega_2)$                                                                                                                                                                                                                  |
| Phasor form                             | $\mathbf{P}_p^{(2)2\omega_0} = \frac{\varepsilon_0}{2} \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}$ $\mathbf{P}_p^{(2)0} = \frac{\varepsilon_0}{2} \left[ \chi^{(2)}(0; -\omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0*} \right]$ $\mathbf{P}_p^{(2)\omega_{03}} = \varepsilon_0 [\chi^{(2)}(-\omega_{03}; \omega_{01}, \omega_{02})] : \mathbf{E}_p^{\omega_{01}} \mathbf{E}_p^{\omega_{02}*}$ |

phasor form. Note that although not explicitly written, the phasor forms of the polarizations and the fields can be slowly varying functions of time,  $t$ , as in the case of linear polarization (see Table 1).

For completeness as well as for future reference, we now write down the corresponding relations for the third-order polarization  $\mathbf{P}^{(3)}$  in the time and frequency domains, as well as its phasor forms. Note that in the time domain, we can write

$$\begin{aligned} \mathbf{P}^{(3)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 d\tau_3 \mathbf{X}^{(3)}(\tau_1, \tau_2, \tau_3) \\ &: \mathbf{E}(t - \tau_1) \mathbf{E}(t - \tau_2) \mathbf{E}(t - \tau_3). \end{aligned} \quad (3-25)$$

Using the frequency domain representation,

$$\begin{aligned} \mathbf{P}^{(3)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \chi^{(3)} \\ &\times (-\omega_1 - \omega_2 - \omega_3; \omega_1, \omega_2, \omega_3) \\ &: \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \mathbf{E}(\omega_3) \exp j\omega t \delta(\omega - \omega_1 - \omega_2 - \omega_3) \end{aligned} \quad (3-26)$$

where

$$\begin{aligned} \chi^{(3)}(-\omega_1 - \omega_2 - \omega_3; \omega_1, \omega_2, \omega_3) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \mathbf{X}^{(3)} \\ &\times (\tau_1, \tau_2, \tau_3) \cdot \exp - j(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3) \end{aligned} \quad (3-27)$$

is the *cubic susceptibility* tensor. Now, once again, as in the case of the second-order polarization, the phasor forms of the third-order polarization depend on the nature of the third-order process involved in the interaction. This determines not only the right  $\chi^{(3)}$ 's but also the correct value of the numerical factor multiplying the nonlinear susceptibilities and the optical fields. Once again, we have to use the invariance property of the nonlinear (in this case, third order) susceptibilities, viz.,  $\chi_{ijk}^{(3)}(-\omega_1 - \omega_2 - \omega_3; \omega_1, \omega_2, \omega_3) = \chi_{ikj}^{(3)}(-\omega_1 - \omega_2 - \omega_3; \omega_2, \omega_1, \omega_3)$ . For instance, in the case of *third harmonic generation*, the requisite third-order polarization phasor would be

$$\mathbf{P}_p^{(3)3\omega_0} = \frac{\epsilon_0}{2} \chi^{(3)}(-3\omega_0; \omega_0, \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}. \quad (3-28)$$

In the case of the *intensity-dependent refractive index* or the optical *Kerr effect* which explains *self-focusing*, *self-* and *cross-phase modulation*, *optical bistability*, and *phase conjugation*, the third-order polarization is

$$\mathbf{P}_p^{(3)\omega_0} = \frac{3\epsilon_0}{4} \chi^{(3)}(-\omega_0; \omega_0, -\omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0*} \quad (3-29)$$

which is in agreement with our simplified analysis in Sec. 2. Finally, in the case of *d.c.-induced second harmonic generation*, also facilitated by the third-order nonlinear susceptibility through the interaction  $0 + \omega_0 + \omega_0 \rightarrow 2\omega_0$ , the third-order polarization takes the form

$$\mathbf{P}_p^{(3)2\omega_0} = \frac{3\epsilon_0}{2} \chi^{(3)}(-2\omega_0; 0, \omega_0, \omega_0) : \mathbf{E}_{d.c.} \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0} \quad (3-30)$$

where  $\mathbf{E}_{d.c.}$  is the applied d.c. field. Table 3 summarizes the definitions of the third-order polarization in the time and frequency domains and lists the relationships between some of the polarizations and the optical fields in phasor form. Note that once again, the phasor forms of the polarizations and the fields can be slowly varying functions of time  $t$ . For a more comprehensive list, we refer readers to Butcher and Cotter (1990).

### 3.3 Discussion

In summary, based on our exposure to the first-, second-, and third-order polarizations in the previous two subsections, we can express the polarization  $\mathbf{P}(t)$ , in general, in the time domain as

**Table 3** Time, Frequency, and Phasor Domain Formulation Showing Dependence of the Third Polarization on the Optical Field

| Regime                                  | Formula                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|-----------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Time domain<br>formulation—general      | $\mathbf{P}^{(3)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 d\tau_3 \mathbf{X}^{(3)}(\tau_1, \tau_2, \tau_3)$ $: \mathbf{E}(t - \tau_1) \mathbf{E}(t - \tau_2) \mathbf{E}(t - \tau_3)$                                                                                                                                                                                                                                                  |
| Frequency domain<br>formulation—general | $\mathbf{P}^{(2)}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3$ $\chi^{(3)}(-\omega_1 - \omega_2 - \omega_3; \omega_1, \omega_2, \omega_3)$ $: \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \mathbf{E}(\omega_3)$ $\times \exp j\omega t \delta(\omega - \omega_1 - \omega_2 - \omega_3)$                                                                                                                                       |
| Phasor form                             | $\mathbf{P}_p^{(3)3\omega_0} = \frac{\varepsilon_0}{2} \chi^{(3)}(-3\omega_0; \omega_0, \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}$ $\mathbf{P}_p^{(3)\omega_0} = \frac{3\varepsilon_0}{4} \chi^{(3)}(-\omega_0; \omega_0, -\omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}$ $\mathbf{P}_p^{(3)2\omega_0} = \frac{3\varepsilon_0}{2} \chi^{(3)}(-2\omega_0; 0, \omega_0, \omega_0) : \mathbf{E}_{d.c.} \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}$ |

$$\mathbf{P}(t) = \mathbf{P}^{(1)}(t) + \mathbf{P}^{(2)}(t) + \mathbf{P}^{(3)}(t) + \dots = \mathbf{P}^L(t) + \mathbf{P}^{NL}(t) \quad (3-31)$$

where  $\mathbf{P}^L(t) = \mathbf{P}^{(1)}(t)$  and  $\mathbf{P}^{NL}(t) = \mathbf{P}^{(2)}(t) + \mathbf{P}^{(3)}(t) + \dots$ . We have also derived the slowly varying phasor forms of the polarization in the previous subsection. Following Shen (1984), we can again express the total polarization in the form of Eq. (3-31) where

$$\mathbf{P}^L(t) = \text{Re} \left[ \sum_i \mathbf{P}_p^{(1)\omega_{0i}}(t) \exp j\omega_{0i}t \right], \quad (3-32a)$$

$$\mathbf{P}_p^{(1)\omega_{0i}}(t) = \varepsilon_0 \chi^{(1)}(-\omega_{0i}; \omega_{0i}) \mathbf{E}_p^{\omega_{0i}}(t); \quad (3-32b)$$

$$\mathbf{P}^{NL}(t) = \text{Re} \left[ \sum_j \mathbf{P}_p^{(2)\omega_{0j}}(t) \exp j\omega_{0j}t + \sum_k \mathbf{P}_p^{(3)\omega_{0k}}(t) \exp j\omega_{0k}t \right] \quad (3-33)$$

where  $\mathbf{P}_p^{(2)\omega_{0j}}(t)$  and  $\mathbf{P}_p^{(3)\omega_{0k}}(t)$  may have the forms as shown in Tables 2 and 3, respectively. In writing the above relations, we have assumed that the total optical field  $\mathbf{E}(t)$  is expressible as

$$\mathbf{E}(t) = \text{Re} \left[ \sum_i \mathbf{E}_p^{\omega_{0i}}(t) \exp j\omega_{0i}t \right]. \quad (3-34)$$

We would wish to point out at this point that the general formulation of a problem in nonlinear optics is extremely complicated. Suppose, for instance, that we have an optical wave at a frequency  $\omega_0$  entering a nonlinear medium which has quadratic (second order) and cubic (third order) nonlinearities. Then the quadratic nonlinearity will first generate the second harmonic at  $2\omega_0$ , which may interact again with the fundamental at  $\omega_0$  to contribute at  $\omega_0 (= 2\omega_0 - \omega_0)$  and at  $3\omega_0 (= 2\omega_0 + \omega_0)$ . The third harmonic can again interact to produce higher harmonics. At the same time, the cubic nonlinearity, using the same argument, will generate all odd-order harmonics, besides contributing to the fundamental through the interaction  $\omega_0 + \omega_0 - \omega_0 \rightarrow \omega_0$ . Thus, in order to analyze the general problem, we would need to a priori write the total field in the interacting medium in the form of Eq. (3-34) where  $\omega_{0i} = i\omega_0$  and compute the virtually infinite frequency components of the nonlinear polarization. To keep matters under control, therefore, one often terminates the series of frequencies at the highest frequency of interest and restricts, in most cases, the analysis to one type of nonlinearity, viz., quadratic or cubic. For instance, if we are interested in studying the second harmonic generation due to a quadratic nonlinearity, we will only consider two frequency components of the optical field, viz., around  $\omega_0$  and  $2\omega_0$ , and, similarly, the corresponding components of the nonlinear polarization. Hence we will write

$$\begin{aligned} \mathbf{E}(t) &= \text{Re}[\mathbf{E}_p^{\omega_0} \exp j\omega_0 t + \mathbf{E}_p^{2\omega_0} \exp 2j\omega_0 t] \\ \mathbf{P}^{\text{NL}}(t) &= \text{Re}[\mathbf{P}_p^{(2)\omega_0} \exp j\omega_0 t + \mathbf{P}_p^{(2)2\omega_0} \exp 2j\omega_0 t] \end{aligned} \quad (3-35)$$

where

$$\begin{aligned} \mathbf{P}_p^{(2)\omega_0} &= \epsilon_0 \chi^{(2)}(-\omega_0; 2\omega_0, \omega_0) : \mathbf{E}_p^{2\omega_0} \mathbf{E}_p^{\omega_0} * \\ \mathbf{P}_p^{(2)2\omega_0} &= (\epsilon_0/2) \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) : \mathbf{E}_p^{\omega_0} \mathbf{E}_p^{\omega_0}. \end{aligned} \quad (3-36)$$

Likewise, if we wish to analyze self-focusing or phase conjugation, we only concern ourselves with analysis around the frequency  $\omega_0$  of the incident optical wave. In this case, we will take

$$\begin{aligned} \mathbf{E}(t) &= \text{Re}[\mathbf{E}_p^{\omega_0} \exp j\omega_0 t] \\ \mathbf{P}^{\text{NL}}(t) &= \text{Re}[\mathbf{P}_p^{(3)\omega_0} \exp j\omega_0 t]. \end{aligned} \quad (3-37)$$

A word of caution at this point: unfortunately, the nonlinear optics literature contains a big jumble of different conventions; in some cases, in the same book or paper. In some cases, for instance, one omits a factor of  $(1/2)$  which must be written when one takes the real part of a complex quantity. Another reason for confusion is the definition of nonlinear polarization which (a) may or may not contain the  $\varepsilon_0$  and (b) may have numerical factors in front on the phasor polarization at a given frequency incorporated in a newly defined form of the nonlinear susceptibility. This, to quote from Butcher and Cotter (1991), “has a number of disadvantages: it becomes confusing to compare the magnitudes of the susceptibilities for different purposes, and also the redefined susceptibility undergoes discontinuous jumps in magnitude when any of the frequency arguments are allowed to become or tend to zero.” Another point to remember is that since we are mostly talking in terms of propagating fields, two frequencies, e.g.,  $\omega_0$  and  $2\omega_0$ , may have different phase velocities due to dispersion. We have seen a simple reason for the origin of dispersion in Sec. 1. In this case, the time domain polarization has to be decomposed in terms of the different propagating wave components of the optical field. Also, the various  $\mathbf{X}$ ’s become functions of space and the  $\chi$ ’s become functions of spatial frequencies. We will talk more on this in connection with second harmonic generation in the next chapter. We remind readers that dispersion is not important when dealing with effects such as self-focusing, phase conjugation, etc. since we restrict ourselves only to one operating frequency.

Suppose, now, that we would like to study, for instance, the spatial evolution of a certain frequency component of the optical field during its propagation through a nonlinear medium. Typical examples would be the study of the growth of the second harmonic in a quadratically nonlinear material or the change in the beam profile during self-focusing in a cubically nonlinear medium. We can achieve this by using the wave equation, derived from Maxwell’s equations, as a starting point. Recall from Chap. 1 that in a homogenous medium, free of charges and currents, which may be anisotropic and nonlinear, an intermediate step during the derivation of the wave equation takes the form

$$\partial^2 \mathbf{D} / \partial t^2 - (1/\mu_0) \nabla^2 \mathbf{E} = 0. \quad (3-38)$$

To recast the above equation in a more standard form, we first set  $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \mathbf{P}^L + \mathbf{P}^{NL}$ . Also, as long as we restrict ourselves to fields and polarizations that can be represented in terms of slowly time-varying phasors, we can use Eqs. (3-32a), (3-32b), and (3-34) to write  $\mathbf{P}^L(t) = \varepsilon_0 \chi^{(1)} \mathbf{E}(t)$  where it is understood the  $\chi^{(1)}$  will take on values consistent with the pertinent frequency component of the  $\mathbf{E}$  field in question. In most cases, we will

take  $\chi^{(1)}$  to be independent of frequency, for the sake of simplicity. Note that if we define  $\varepsilon = \varepsilon_0(1 + \chi^{(1)})$ , we can recast Eq. (3-38) in the form

$$\varepsilon \partial^2 \mathbf{E} / \partial t^2 - (1/\mu_0) \nabla^2 \mathbf{E} = -\partial^2 \mathbf{P}^{\text{NL}} / \partial t^2. \quad (3-39)$$

If we make the additional simplification that the  $\chi^{(1)}$  is a scalar, then we can rewrite Eq. (3-39) as

$$\partial^2 \mathbf{E} / \partial t^2 - (1/\mu_0 \varepsilon) \nabla^2 \mathbf{E} = -(1/\varepsilon) \partial^2 \mathbf{P}^{\text{NL}} / \partial t^2. \quad (3-40)$$

In many instances in the book, we will use Eq. (3-40) as one of our starting points during the analysis of various nonlinear processes, unless otherwise stated.

#### 4 AN ALTERNATIVE APPROACH

Thus far, we have presented a rather detailed treatment of the modeling of optical nonlinearities in terms of the nonlinear polarization and the second- and third-order nonlinear susceptibilities. We have also formulated the nonlinear wave equation for the optical field  $\mathbf{E}$  where the effect from nonlinear polarization  $\mathbf{P}^{\text{NL}}$  can be treated as a source term in the system. In this section, we present an alternate approach to the modeling of nonlinear propagation which has its origin in fluid mechanics.

We begin by postulating that in a nonlinear medium, the nonlinearity can be attributed to the dependence of the phase velocity on the amplitude of the propagating wave, unlike the linear case. Recall from Chap. 1 that a PDE for a scalar wave function  $\psi(z, t)$  of the form

$$\partial^2 \psi / \partial t^2 - v^2 \partial^2 \psi / \partial z^2 = 0 \quad (4-1)$$

models bidirectional wave propagation in a linear nondispersive isotropic medium. However, we can model unidirectional propagation (viz., along  $+z$ ) by an equation of first order in time of the form

$$\partial \psi / \partial t + v \partial \psi / \partial z = 0, \quad v > 0. \quad (4-2)$$

We postulate that in a nonlinear medium, the phase velocity  $v$  is modified to  $v'$  according to

$$v' = v(1 + \beta_2 \psi + \beta_3 \psi^2 + \dots), \quad (4-3)$$

where  $\beta_2$  and  $\beta_3$  are constants (Whitham, 1974). Thus, in the nonlinear regime, Eq. (4-2) is modified, in light of Eq. (4-3), to

$$\partial \psi / \partial t + v(1 + \beta_2 \psi + \beta_3 \psi^2) \partial \psi / \partial z = 0 \quad (4-4a)$$

or

$$\partial\psi/\partial t + v\partial\psi/\partial z + (v\beta_2/2)\partial\psi^2/\partial z + (v\beta_3/3)\partial\psi^3/\partial z = 0. \quad (4-4b)$$

Note that the above equation is a nonlinear PDE in  $\psi(z,t)$ , containing quadratic and cubic powers of  $\psi$ . The coefficients  $\beta_2$  and  $\beta_3$  are called the *quadratic* and *cubic nonlinear coefficients*, respectively.

The above model of nonlinear propagation is derived from fluid mechanics. It is interesting how the same overall effect is perceived from different angles in various disciplines. For instance, in fluid mechanics, the nonlinearity, e.g.,  $\beta_2$ , is responsible for shock formation during fluid flow due to steepening of the (baseband) wave profile, as shown in Fig. 3. This happens because the parts with smaller amplitudes, for  $\beta_2 > 0$ , lead to a point in time where the right edge of the waveform develops infinite steepness or *shock*. In optics, the quadratic nonlinearity, like  $\chi^{(2)}$ , is responsible for second harmonic generation, as we have stated before. The cubic nonlinearity is responsible for effects such as self-focusing, bistability, phase conjugation, and soliton propagation, to name a few.

The simple model of nonlinearity, described above, cannot, however, describe wave propagation in higher dimensions  $(x,y,z)$ , necessitating the need for a higher-order PDE. As an illustrative example, we demonstrate how to derive such an equation for  $\beta_2 \neq 0$  and  $\beta_3 = 0$  (Korpel and Banerjee, 1984). First, we differentiate Eq. (4-4b) with respect to  $t$  to get

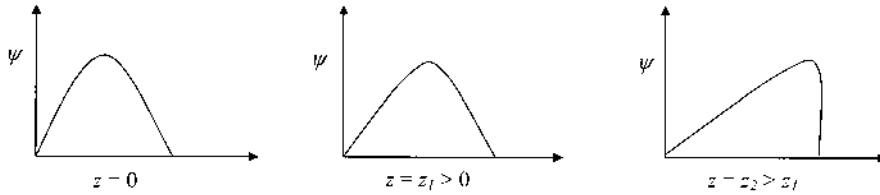
$$\partial^2\psi/\partial t^2 + v\partial/\partial z[\partial\psi/\partial t] + (v\beta_2)\partial/\partial z[\psi\partial\psi/\partial t] = 0. \quad (4-5)$$

Now we substitute for  $\partial\psi/\partial t$  from Eq. (4-4a) and neglect higher-order terms in  $\beta_2$  (assuming weak nonlinearity) to obtain

$$\partial^2\psi/\partial t^2 - v\partial^2\psi/\partial z^2 \approx v^2\beta_2\partial^2\psi^2/\partial z^2. \quad (4-6)$$

Because the nonlinearity has been assumed to be weak, we can replace  $z$  by  $vt$  on the RHS of the above equation and write

$$\partial^2\psi/\partial t^2 - v^2\partial^2\psi/\partial z^2 \approx \beta_2\partial^2\psi^2/\partial t^2. \quad (4-7)$$



**Figure 3** Evolution of shock during propagation of a pulse, assuming  $\beta_2 > 0$ . The horizontal axis  $\tau = t - z/v$  is a moving frame of reference.

For more general case, when  $\beta_2 \neq 0$  and  $\beta_3 \neq 0$ , a similar manipulation gives

$$\partial^2 \psi / \partial t^2 - v^2 \partial^2 \psi / \partial z^2 \approx \beta_2 \partial^2 \psi^2 / \partial t^2 + (2\beta_3/3) \partial^2 \psi^3 / \partial t^2. \quad (4-8)$$

We can extend Eq. (4.8) to higher dimensions by replacing  $\partial^2 / \partial z^2$  by the Laplacian  $\nabla^2$ :

$$\partial^2 \psi / \partial t^2 - v^2 \nabla^2 \psi \approx \beta_2 \partial^2 \psi^2 / \partial t^2 + (2\beta_3/3) \partial^2 \psi^3 / \partial t^2. \quad (4-9)$$

Here, too, notice that  $\beta_2$  and  $\beta_3$  appear with the terms involving  $\psi^2$  and  $\psi^3$ , respectively, justifying the names quadratic and cubic nonlinearity coefficients given previously.

We are now in a position to compare the results from the simplified approach to those we obtained in the previous subsection from a more rigorous treatment of polarization. Note that the form of Eq. (4-9) is similar to Eq. (3-40), derived in Sec. 3.3 if we identify  $\psi$  to be similar to  $\mathbf{E}$ . The quantity  $1/\mu_0\epsilon$  in Eq. (3-40) is then similar to  $v^2$ . Also, the nonlinear polarization  $\mathbf{P}^{\text{NL}}$  can be compared with  $-\epsilon\beta_2\psi^2 - (2/3)\epsilon\beta_3\psi^3$ . Indeed, note that the different components of the nonlinear polarization come from squaring and cubing the optical field  $\mathbf{E}$ , which eventually leads to the various relations between the second- and third-order nonlinear polarization phasors and the optical field phasors, as shown in Tables 2 and 3, respectively. We will get into more details about the relationships between the simplistically defined nonlinear coefficients  $\beta_2$  and  $\beta_3$  and the pertinent second- and third-order nonlinear susceptibilities as well as the nonlinear refractive index coefficient in the chapters to follow.

## 5 SUMMARY

In this chapter, we have introduced a simple physical model for the origin of optical nonlinearities based on the forces acting on an electron which is separated from the nucleus by an externally applied electric (or optical) field. We also named some of the physical affects of the induced nonlinear polarization, viz., second harmonic generation, self-focusing, phase conjugation, optical bistability, etc. We then discussed the rigorous mathematical modeling of optical nonlinearities, both in the time and frequency domains, and in the case when the optical fields as well as the nonlinear polarization are expressible in terms of slowly time-varying phasors. We derived a nonlinear extension of the wave equation, showing the role of the induced nonlinear polarization as a driving or source term. Finally, we took a more simplistic view of nonlinear change in the phase velocity of a wave. This helped us derive a nonlinear wave equation which is structurally similar to the one that was previously derived more rigorously. The aim of this exercise was to try to

develop as simple a mathematical model as possible to study various second- and third-order nonlinear (optical) effects.

## 6 PROBLEMS

1. Write down the governing differential equation for the electron displacement with respect to the nucleus using the nonlinear oscillator model in the presence of damping and a second-order nonlinearity when there is a driving optical field  $E_0 \cos \omega_0 t$ . Hence find an expression for the nonlinear susceptibility  $\chi^{(2)}(\omega_0)$  responsible for second harmonic generation.
2. Repeat the above problem for the case when the driving optical field is  $E_1 \cos \omega_1 t + E_2 \cos \omega_2 t$ . In this case, find an expression for the nonlinear susceptibility  $\chi^{(2)}(\omega_1 + \omega_2)$  responsible for sum frequency generation.
3. Write down an expression for the  $n$ -th-order polarization  $\mathbf{P}^{(n)}(t)$  in the time domain formulation. Recast this in the frequency domain in terms of the spectra of the optical fields and the  $n$ th-order nonlinear susceptibility tensor.
4. In the general case of four-wave mixing, i.e.,  $\omega_1 + \omega_2 + \omega_3 \rightarrow \omega_4$ , derive an expression for the phasor form of the polarization in terms of the phasor forms of the participating optical fields and the requisite third-order nonlinear susceptibility tensor.
5. Consider the simplified model of unidirectional nonlinear propagation comprising a PDE of first order in time. Assuming only a cubic nonlinearity coefficient and that the wave function  $\psi$  comprises an envelope  $\psi_e$  on a propagating carrier at frequency  $\omega_0$ , find the PDE governing the evolution of the envelope. Neglect the generation of the third harmonic. By making a suitable change of independent variables, solve this equation in terms of the value of  $\psi$  at  $z=0$ .
6. Solve the simplified model of unidirectional nonlinear propagation as in Problem 5 by assuming only a quadratic nonlinearity coefficient and an initial condition  $\psi(z,0) = A \sin Kz$ .

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# 3

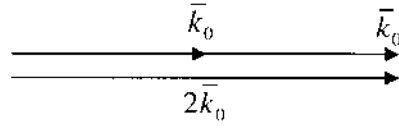
## Second Harmonic Generation

In Chap. 2, we have discussed at length the nonlinear polarization in a material and derived the nonlinear wave equation rigorously. We have also shown that a scalar nonlinear wave equation can be heuristically derived from the concept that the phase velocity of a wave is modified in the presence of nonlinearities. In this chapter, we will employ the more rigorous approach to investigate second harmonic generation (SHG) in quadratically nonlinear materials. We examine the effect of linear phase mismatch, and of the beam profile and crystal anisotropy on the conversion efficiency. We would like to point out that, in general, both quadratic and cubic nonlinearities may exist in a material. It turns out that “cascaded” quadratic nonlinearities can, in some cases, simulate an effective cubic nonlinearity. Spatial solitons commonly exist in a cubically nonlinear material, as will be shown in the following chapter.

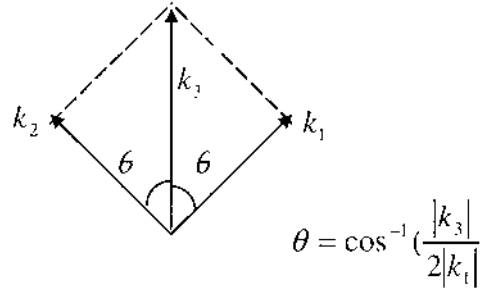
### 1 THE PHYSICS OF SECOND HARMONIC GENERATION

In any kind of wave interaction process, there are three conditions that need to be satisfied for an efficient conversion of energy from one wave to another. These are (a) energy conservation, (b) momentum conservation, and (c) dispersion relation (Banerjee and Poon, 1991). Energy conservation states that  $\hbar\omega_0 + \hbar\omega_0 \rightarrow \hbar(2\omega_0)$ , where  $\hbar = h/2\pi$ , with  $h$  being Planck’s constant, or, equivalently,  $\omega_0 + \omega_0 \rightarrow 2\omega_0$ . The first relation relates the energies of the participating photons. Once photons at energy  $2\omega_0$  have been created, they can interact with the fundamental (photons at  $\omega_0$ ) to downconvert energy from  $2\omega_0$  to  $\omega_0$ , through the relation  $\hbar(2\omega_0) - \hbar(\omega_0) \rightarrow \hbar(\omega_0)$ ,  $2\omega_0 - \omega_0 \rightarrow \omega_0$ .

In a similar fashion, the momentum conservation should be satisfied. Referring to Fig. 1a, note that the momentum relation gives  $\hbar\mathbf{k}_0 + \hbar\mathbf{k}_0 \rightarrow \hbar(2\mathbf{k}_0)$ , or, equivalently,  $\mathbf{k}_0 + \mathbf{k}_0 \rightarrow 2\mathbf{k}_0$  for collinear upconversion. The first relation relates the momenta of the photons—similarly,  $2\mathbf{k}_0 - \mathbf{k}_0 \rightarrow \mathbf{k}_0$  for the downconversion process. Although this picture is true for the case when there



(a)



$$|k_1| = |k_2|$$

$$\omega_1 = \omega_2 = \frac{\omega_3}{2} = \omega_0$$

(b)

**Figure 1** Resonant triad wavevector diagram representing SHG for the case of (a) no dispersion, (b) dispersion.

is no dispersion, it turns out that if  $\mathbf{k}(2\omega_0) \neq 2\mathbf{k}(\omega_0)$ , the situation is quite different. Thus for the dispersive case, if the waves are free to move in higher dimensions and  $|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}(\omega_0)| > 1/2|\mathbf{k}_3|$ , where  $|\mathbf{k}_3| = |\mathbf{k}(2\omega_0)|$ , the momentum triangle looks as in Fig. 1b. In this case, the angle between the fundamental harmonic and the second harmonic is given by:

$$\theta = \cos^{-1} \frac{|\mathbf{k}_3|}{2|\mathbf{k}_1|} = \triangleq \cos^{-1}(k_3/2k_1) \quad (1-1)$$

where  $k_{1,2,3} = |\mathbf{k}_{1,2,3}|$  and has been observed during interaction of, for instance, water waves (Banerjee and Korpel, 1982).

If waves are, however, restricted to move in only one dimension, we may get spatial beating of the waves with a linear beating period:

$$\Lambda = 2\pi/|(k_2 - 2k_1)|, \quad (1-2)$$

where we have now renamed  $k_2 = k(2\omega_0)$  and  $k_1 = k(\omega_0)$ . In optics, this phenomenon is referred to as the *phase mismatched case* (Armstrong et al., 1962). Sometimes, phase matching conditions, which ensure maximum conversion, can be achieved by ensuring that  $|\mathbf{k}_2| = 2|\mathbf{k}_1|$ , through, for instance, having the fundamental as an ordinary wave in an anisotropic crystal, with the second harmonic being generated as an extraordinary wave. Such an interaction is called a *Type 1 interaction*, or an *oo-e* interaction. The “o” stands for the ordinary polarization of the fundamental waves, whereas the “e” stands for the extraordinary polarization of the second harmonic. In other cases, an external grating with a grating vector of magnitude equal to  $|\mathbf{k}_2 - \mathbf{k}_1|$  may be fabricated or stored optically in the interaction region, helping to realize quasi-phase matching conditions (for instance, see Shen, 1984).

The presence of a *cubic nonlinearity* in the material can also affect the generation of the second harmonic. Cubic nonlinearities are responsible for *self-focusing* and *self-defocusing* of optical beams, and also give rise to interesting effects such as *optical bistability*, *phase conjugation*, and *optical spatial and temporal solitons*. This is discussed in later chapters. The presence of a cubic nonlinearity (over and above the quadratic nonlinearity) can give rise to additional interactions that can be summarized through the relations:

$$\begin{aligned} \omega_0 + \omega_0 - \omega_0 &\rightarrow \omega_0 & 2\omega_0 + \omega_0 - \omega_0 &\rightarrow 2\omega_0 \\ \omega_0 + 2\omega_0 - 2\omega_0 &\rightarrow \omega_0 & 2\omega_0 + 2\omega_0 - 2\omega_0 &\rightarrow 2\omega_0. \end{aligned} \quad (1-3)$$

In some cases, the cubic nonlinearity in the material has been used to achieve *quasi-phase matching* conditions even in the presence of a linear phase mismatch (Razumikhina et al., 1984; Choe et al., 1991). In other situations, the cubic nonlinearity can be exploited (in the absence of a formal quadratic nonlinearity) to generate the second harmonic in the presence of a strong DC bias:  $\omega_0 + \omega_0 \pm 0 \rightarrow 2\omega_0$ ;  $2\omega_0 - \omega_0 \pm 0 \rightarrow \omega_0$ , where the “0” stands for the frequency of the DC bias. This is one explanation put forth to explain SHG in poled optical fibers (Osterberg and Margulis, 1986).

The simplified discussion presented above is valid for participating plane waves. On the other hand, in some experimental situations, the fundamental is often focused into the interaction region to achieve greater intensity levels that increase SHG (for instance, see Boyd, 1992). Although the fundamental and second harmonic profiles remain approximately Gaussian

for low conversion efficiencies, their profiles may significantly deviate from Gaussian during depletion of the fundamental.

## 2 SHG IN CRYSTALS—MATHEMATICAL FORMULATION

We start with Eq. (3-40) in Chap. 2 in the presence of a quadratic nonlinearity:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - v^2 \nabla^2 \mathbf{E} = -\frac{1}{\varepsilon} \frac{\partial^2 \mathbf{P}^{(2)}}{\partial t^2} \quad (2-1)$$

where  $\mathbf{P}^{(2)}$  represents the nonlinear second-order polarization and  $v$  is the phase velocity, to be made more precise below. Now, putting:

$$\mathbf{E} = \frac{1}{2} [\mathbf{E}_e^{\omega_0} \exp j(\omega_0 t - k_1 z) + \mathbf{E}_e^{2\omega_0} \exp j(2\omega_0 t - k_2 z) + \text{c.c.}] \quad (2-2)$$

we can write

$$\begin{aligned} \frac{\partial^2 \mathbf{E}}{\partial t^2} = & -\frac{1}{2} [\omega_0^2 \mathbf{E}_e^{\omega_0} \exp j(\omega_0 t - k_1 z) \\ & + 4\omega_0^2 \mathbf{E}_e^{2\omega_0} \exp j(2\omega_0 t - k_2 z) + \text{c.c.}] \end{aligned} \quad (2-3a)$$

$$\begin{aligned} \nabla^2 \mathbf{E} \approx & -\frac{1}{2} \left[ \left( 2jk_1 \frac{\partial \mathbf{E}_e^{\omega_0}}{\partial z} + k_1^2 \mathbf{E}_e^{\omega_0} \right) \exp j(\omega_0 t - k_1 z) \right. \\ & \left. + \left( 2jk_2 \frac{\partial \mathbf{E}_e^{2\omega_0}}{\partial z} + k_2^2 \mathbf{E}_e^{2\omega_0} \right) \exp j(2\omega_0 t - k_2 z) \right]. \end{aligned} \quad (2-3b)$$

Substituting Eqs. (2-3a) and (2-3b) into Eq. (2-1) and collecting terms around  $\omega_0, 2\omega_0$ , we get the two following equations:

$$\begin{aligned} & -\frac{\omega_0^2}{2} \mathbf{E}_e^{\omega_0} + \frac{v_1^2}{2} \left( 2jk_1 \frac{\partial \mathbf{E}_e^{\omega_0}}{\partial z} + k_1^2 \mathbf{E}_e^{\omega_0} \right) \\ & = \frac{\omega_0^2}{2\varepsilon_1} \varepsilon_0 \chi^{(2)}(-\omega_0; 2\omega_0, \omega_0) \mathbf{E}_e^{\omega_0*} \mathbf{E}_e^{2\omega_0} e^{-j(k_2 - 2k_1)z} \end{aligned} \quad (2-4a)$$

and

$$\begin{aligned} & -2\frac{\omega_0^2}{2} \mathbf{E}_e^{\omega_0} + \frac{v_2^2}{2} \left( 2jk_2 \frac{\partial \mathbf{E}_e^{\omega_0}}{\partial z} + k_2^2 \mathbf{E}_e^{\omega_0} \right) \\ & = \frac{\omega_0^2}{2\varepsilon_1} \varepsilon_0 \chi^{(2)}(-2\omega_0; \omega_0, \omega_0) \mathbf{E}_e^{\omega_0} \mathbf{E}_e^{\omega_0} e^{-j(k_2 - 2k_1)z}. \end{aligned} \quad (2-4b)$$

In deriving Eqs. (2-4a) and (2-4b), we have replaced the  $v$  in Eq. (2-1) by  $v_1$  and  $v_2$ , respectively, and the  $\varepsilon$  by  $\varepsilon_1$  and  $\varepsilon_2$ , respectively, to incorporate the effect(s)

of linear dispersion on the phase velocity and the permittivity. Note that this is reasonable in view of the fact that we are only interested in two discrete frequencies. Formal methods to include dispersion for a continuum of frequencies will be dealt with in a later chapter. Upon defining  $\Delta k = k_2 - 2k_1$ , Eqs. (2-4a) and (2-4b) can be simplified to give:

$$jv_1^2 k_1 \frac{d\mathbf{E}_e^{\omega_0}}{dz} = \frac{\omega_0^2 \varepsilon_0}{2\varepsilon_1} \chi^{(2)}(-\omega_0; 2\omega_0; -\omega_0) \mathbf{E}_e^{2\omega_0} \mathbf{E}_e^{\omega_0*} e^{-j\Delta k z} \quad (2-5a)$$

$$jv_2^2 k_2 \frac{d\mathbf{E}_e^{\omega_0}}{dz} = \frac{\omega_0^2 \varepsilon_0}{2\varepsilon_2} \chi^{(2)}(-2\omega_0; \omega_0; \omega_0) \mathbf{E}_e^{\omega_0} \mathbf{E}_e^{\omega_0} e^{+j\Delta k z}. \quad (2-5b)$$

Now,

$$\frac{\varepsilon_0}{2\varepsilon_1} \cdot \frac{1}{v_1^2 k_1} = \frac{1}{2c^2 k_1}, \quad \frac{\varepsilon_0}{2\varepsilon_2} \cdot \frac{1}{v_2^2 k_2} = \frac{1}{2c^2 k_2}. \quad (2-6)$$

Using Eq. (2-6) in Eqs. (2-5a) and (2-5b), we get:

$$\frac{d\mathbf{E}_e^{\omega_0}}{dz} = \frac{-j\omega_0^2}{2c^2 k_1} \chi^{(2)}(-\omega_0; 2\omega_0; -\omega_0) \mathbf{E}_e^{2\omega_0} \mathbf{E}_e^{\omega_0*} e^{-j\Delta k z}, \quad (2-7a)$$

$$\frac{d\mathbf{E}_e^{2\omega_0}}{dz} = \frac{-j\omega_0^2}{2c^2 k_1} \chi^{(2)}(-2\omega_0; \omega_0, -\omega_0) \mathbf{E}_e^{\omega_0} e^{j\Delta k z}. \quad (2-7b)$$

## 2.1 Undepleted Fundamental

We assume the optical fields to be  $x$ -polarized for simplicity (i.e.,  $\mathbf{E}_e^{\omega_0, 2\omega_0} = E_e^{\omega_0, 2\omega_0} \mathbf{a}_x$ ). Also, we have  $\mathbf{E}_e^{\omega_0} = \text{constant}$  (undepleted fundamental), so that from Eq. (2-7b):

$$\frac{dE_e^{2\omega_0}}{dz} = \frac{-j\omega_0^2}{c^2 k_2} \chi^{(2)} E_e^{\omega_0 2} e^{j\Delta k z}, \quad (2-8)$$

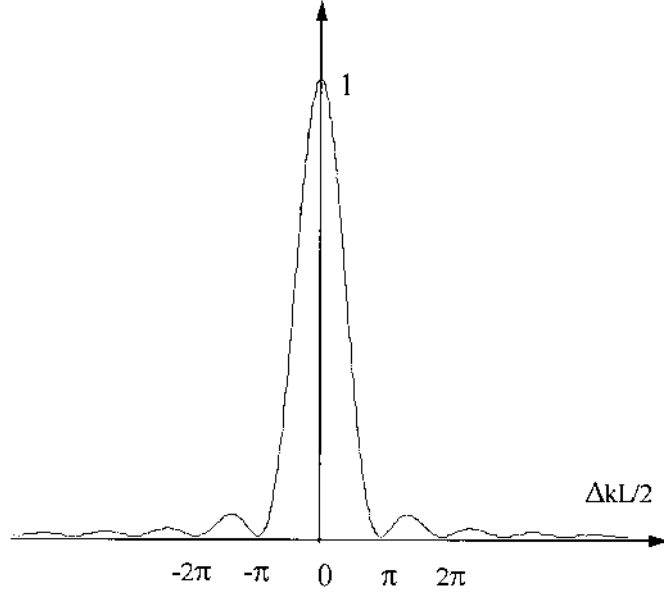
where  $\chi^{(2)} = \chi_{111}^{(2)}(-2\omega_0; \omega_0, \omega_0)$ , which can be readily solved to give:

$$E_e^{2\omega_0} = \frac{j\omega_0^2}{c^2 k_2} \chi^{(2)} E_e^{\omega_0 2} \frac{e^{j\Delta k z} - 1}{j\Delta k}, \quad (2-9a)$$

and hence the second harmonic intensity can be written as:

$$I^{2\omega_0} \propto |E_e^{2\omega_0}|^2 = \left( \frac{\omega_0^2}{c^2 k_2} \chi^{(2)} E_e^{\omega_0 2} \right)^2 L^2 \frac{\sin^2 \frac{\Delta k L}{2}}{\left( \frac{\Delta k L}{2} \right)^2} \quad (2-9b)$$

at the end of the crystal,  $z = L$ , assuming  $\mathbf{E}_e^{2\omega_0} = 0$  at  $z = 0$ . The graph of  $I^{2\omega_0}$  for different  $\Delta k$ 's is shown in Fig. 2.



**Figure 2** Plot of SHG diffraction efficiency as a function of linear phase mismatch.

## 2.2 General Case: Depleted Pump

In this case, from Eqs. (2-7a) and (2-7b), upon again assuming the optical fields to be  $x$ -polarized and substituting:

$$E_e^{\omega_0} = \tilde{E}_e^{\omega_0} e^{-j\Delta kz/2}, \quad E_e^{2\omega_0} = \tilde{E}_e^{2\omega_0} e^{j\Delta kz/2} \quad (2-10)$$

we obtain:

$$\frac{d\tilde{E}_e^{\omega_0}}{dz} \cong \frac{j\omega_0^2}{2c^2k_0} \chi^{(2)} \tilde{E}_e^{2\omega_0} \tilde{E}_e^{\omega_0*} + j\frac{\Delta k}{2} \tilde{E}_e^{\omega_0} \quad (2-11a)$$

$$\frac{d\tilde{E}_e^{2\omega_0}}{dz} \cong \frac{-j\omega_0^2}{2c^2k_0} \chi^{(2)} \tilde{E}_e^{\omega_0 2} - j\frac{\Delta k}{2} \tilde{E}_e^{2\omega_0}. \quad (2-11b)$$

In deriving Eqs. (2-11a) and (2-11b), we have also replaced  $k_1$  by  $k_0$  and  $k_2$  by  $2k_0$  in the nonlinear term on the RHS. Also,  $\chi^{(2)} = \chi_{111}^{(2)}(-2\omega_0; \omega_0, \omega_0) = (1/2)\chi_{111}^{(2)}(-\omega_0; 2\omega_0, -\omega_0)$ . Now, putting

$$\tilde{E}_e^{\omega_0} = a_1 e^{-j\phi_1}, \quad \tilde{E}_e^{2\omega_0} = a_2 e^{-j\phi_2} \quad (2-12)$$

in Eqs. (2-11a) and (2-11b), simplifying and decomposing into real and imaginary parts, we get, after some algebra:

$$\frac{da_1}{dz} = -\frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} a_1 a_2 \sin(\phi_2 - 2\phi_1), \quad (2-13a)$$

$$\frac{da_2}{dz} = -\frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} a_1^2 \sin(\phi_2 - 2\phi_1), \quad (2-13b)$$

$$\frac{d\phi_1}{dz} = -\frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} a_2 \cos(\phi_2 - 2\phi_1) - \frac{\Delta k}{2}, \quad (2-13c)$$

$$\frac{d\phi_2}{dz} = -\frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} \frac{a_1^2}{a_2} \cos(\phi_2 - 2\phi_1) + \frac{\Delta k}{2}. \quad (2-13d)$$

These four equations can be reduced to a set of three by defining a relative phase shift:

$$\phi = \phi_2 - 2\phi_1, \quad (2-14)$$

so that:

$$\frac{da_1}{dz} = -\frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} a_1 a_2 \sin\phi, \quad (2-15a)$$

$$\frac{da_2}{dz} = \frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} a_1^2 \sin\phi, \quad (2-15b)$$

$$\frac{d\phi}{dz} = \frac{3\Delta k}{2} - \frac{\omega_0^2 \chi^{(2)}}{2c^2 k_0} \left(2a_2 - \frac{a_1^2}{a_2}\right) \cos\phi. \quad (2-15c)$$

Note that from Eqs. (2-15a) and (2-15b):

$$a_1 \frac{da_1}{dz} + a_2 \frac{da_2}{dz} = -\frac{1}{2} \left( \frac{da_1^2}{dz} + \frac{da_2^2}{dz} \right) = 0,$$

implying

$$a_1^2 + a_2^2 = \text{constant} = \tilde{E}, \quad (2-16)$$

where  $\tilde{E}$  is the total power in the system. Now using the normalizations:

$$u = a_1 / \tilde{E}^{1/2}, \quad v = a_2 / \tilde{E}^{1/2}, \quad \Delta s = \frac{4c^2 k_0}{\omega_0^2 \chi^{(2)} \tilde{E}^{1/2}} \cdot \frac{3\Delta k}{2}, \quad (2-17)$$

$$\zeta = \frac{\omega_0^2 \chi^{(2)} \tilde{E}^{1/2}}{2c^2 k_0} z,$$

Eqs. (2-15a) and (2-15b) can be rewritten as:

$$\frac{du}{d\zeta} = -uv\sin\phi, \quad (2-18a)$$

$$\frac{dv}{d\zeta} = u^2\sin\phi, \quad (2-18b)$$

$$\frac{d\phi}{d\zeta} = \Delta s - \left(2v - \frac{u^2}{v}\right)\cos\phi. \quad (2-18c)$$

Eqs. (2-18a), (2-18b), and (2-18c) describe the evolution of the fundamental and the second harmonic during SHG.

### 2.3 Perfect Phase Matching: $\Delta s = 0$

In the case of perfect phase matching ( $\Delta s = 0$ ), Eqs. (2-18a), (2-18b), and (2-18c) reduces to:

$$\frac{du}{d\zeta} = -uv\sin\phi, \quad (2-19a)$$

$$\frac{dv}{d\zeta} = u^2\sin\phi \quad (2-19b)$$

$$\frac{d\phi}{d\zeta} = -\left(2v - \frac{u^2}{v}\right)\cos\phi. \quad (2-19c)$$

The general solution procedure is more involved and will be pursued in Sec. 2.4 for the case  $\Delta s \neq 0$ . Suffice here to state that in this case (i.e.,  $\Delta s = 0$ ), the relative phase  $\phi$  remains locked at  $\phi = \pi/2$ . Then, Eq. (2-19c) is automatically satisfied and Eqs. (2-19a) and (2-19b) reduce to:

$$\frac{du}{d\zeta} = -uv; \frac{dv}{d\zeta} = u^2. \quad (2-20)$$

This makes physical sense because the second of the above set of equations predicts initial growth of the second harmonic for small  $\zeta$ . Correspondingly, as soon as the second harmonic is generated, the fundamental starts to deplete, as evident from the first of the equations in Eq. (2-20).

We would like to point out that the condition  $\phi = -\pi/2$  corresponds to the case of *subharmonic generation*, seen often in fluids (Banerjee and Korpel, 1982) but less frequently in optics. In this case, the coupled equations become:

$$\frac{du}{d\zeta} = uv; \frac{du}{d\zeta} = \frac{dv}{d\zeta} = -u^2. \quad (2-21)$$

Hence, with energy initially in  $v$  (i.e.,  $v(0) \neq 0$ ), the  $v$  (now renamed the *fundamental*) starts to deplete whereas  $u$  (now renamed the *subharmonic*)

starts to grow from noise. We will not analyze this here at length but offer it as a problem at the end of the chapter.

Returning to the issue of phase-matched SHG with  $u(0) \neq 0$ ,  $v(0)$ , Eq. (2-20) can be readily decouple using the conservation of energy, which in normalized quantities now simply reads:

$$u^2 + v^2 = 1. \quad (2-22)$$

Substituting this in the second of the equations in Eq. (2-20):

$$\frac{dv}{d\zeta} = 1 - v^2 \quad (2-23)$$

which can be readily integrated to give:

$$v(\zeta) = \tanh \zeta \quad (2-24a)$$

so that, with Eq. (2-22):

$$u(\zeta) = \text{sech} \zeta. \quad (2-24b)$$

In denormalized coordinates, the solutions can be written as:

$$a_1(z) = \tilde{E}^{1/2} \text{sech} \frac{\omega_0^2 \chi^{(2)} \tilde{E}^{1/2}}{2c^2 k_0} z, \quad (2-25a)$$

$$a_2(z) = \tilde{E}^{1/2} \tanh \frac{\omega_0^2 \chi^{(2)} \tilde{E}^{1/2}}{2c^2 k_0} z. \quad (2-25b)$$

The results are plotted in Fig. 3.

## 2.4 The Phase Mismatched Case: $\Delta s \neq 0$

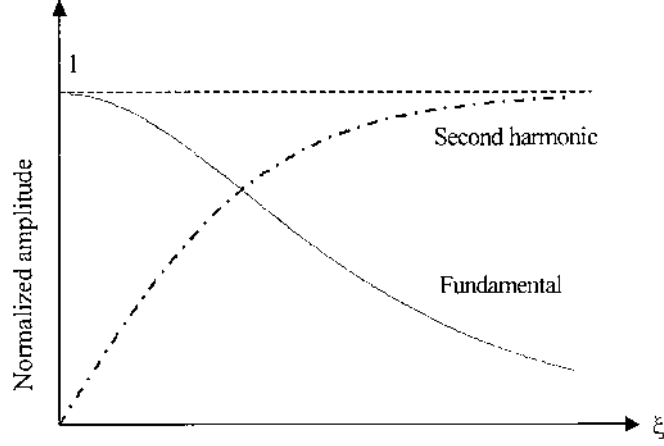
As will be shown during the course of this analysis, there is a periodic exchange of energy between the fundamental and the second harmonic when there is a phase mismatch. This is expected because an oscillatory behavior of the second harmonic was also observed for the undepleted pump case (see Eqs. (2-9a)).

For a formal analysis, we start from Eq. (2.18c) and multiply both sides by  $u^2 v \sin \phi$ . Then:

$$u^2 v \sin \phi \frac{d\phi}{d\zeta} = \Delta s u^2 v \sin \phi - \left( 2v - \frac{u^2}{v} \right) u^2 v \cos \phi \sin \phi. \quad (2-26)$$

Observe now that:

$$\frac{d}{d\zeta} [u^2 v \cos \phi] = 2uv \cos \phi \frac{du}{d\zeta} + u^2 \cos \phi \frac{dv}{d\zeta} - u^2 v \sin \phi \frac{d\phi}{d\zeta}. \quad (2-27)$$



**Figure 3** Theoretically predicted variation of the fundamental and second harmonic due to second order nonlinearity and perfect phase matching.

Upon substituting for  $\frac{du}{d\xi}$ ,  $\frac{dv}{d\xi}$  in the above equation using Eqs. (2-18a) and (2-18b), and noting that  $u^2 v \sin \phi$  in the RHS of Eq. (2-26) can be written as  $(1/2)d(v^2)/d\xi$  using Eq. (2-18b), it follows that:

$$\frac{d}{d\xi} [u^2 v \cos \phi] + \frac{\Delta s}{2} \frac{d(v^2)}{d\xi} = 0. \quad (2-28)$$

Upon integrating, we get:

$$u^2 v \cos \phi + \frac{\Delta s}{2} v^2 = \text{const} = \Gamma_{\Delta s} \equiv \Gamma + \left( \frac{\Delta s}{2} \right) v^2(0) \quad (2-29)$$

in the case where  $v(0)$  may be arbitrary. Substituting the above result in Eq. (2-19b) and after some algebra, we can write the solution of the equation in the form:

$$\xi = \pm \frac{1}{2} \int_{v^2(0)}^{v^2(\xi)} \frac{d(v^2)}{[v^2(1 - v^2)^2 - \{\Gamma - (\Delta s/2)(v^2 - v^2(0))\}^2]^{\frac{1}{2}}}. \quad (2-30)$$

In deriving Eq. (2-30), we have used the conservation of energy relation (2-22).

For the special case  $v(0) = 0$  (i.e., no initial second harmonic):

$$\xi = \pm \frac{1}{2} \int_0^{v^2(\xi)} \frac{d(v^2)}{\left[ v^2(1 - v^2)^2 - \left( \frac{\Delta s}{2} v^2 \right)^2 \right]^{\frac{1}{2}}}. \quad (2-31)$$

The solution of the above equation can be expressed in terms of elliptical functions and are periodic in nature. The period is given by:

$$\Pi_{\xi} = \int_0^{v_b^2} \frac{d(v^2)}{\left[ v^2(1-v^2)^2 - \left( \frac{\Delta s}{2} v^2 \right)^2 \right]^{\frac{1}{2}}} \quad (2-32)$$

where the roots of the part of the denominator in square brackets, that is,

$$D(v^2) = (v^2)^3 - \left( 2 + \left( \frac{\Delta s}{2} \right)^2 \right) (v^2) + v^2 \quad (2-33)$$

can be written as:

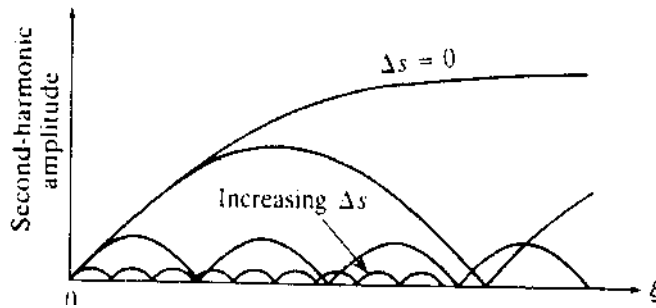
$$v_a^2 = 0; \quad v_b^2 = \frac{1}{v_c^2},$$

$$v_c^2 = \left[ \frac{\Delta s}{4} + \left( 1 + \frac{\Delta s}{4} \right)^{1/2} \right]^2. \quad (2-34)$$

Hence, Eq. (2-32) can be reexpressed as:

$$\zeta = \pm \frac{1}{2} \int_0^{v^2(\zeta)} \frac{d(v^2)}{(v^2)(v^2 - v_b^2)(v^2 - v_c^2)}. \quad (2-35)$$

The exact nature of the solution can be looked up from Abramovitz and Stegun (1965). Typical solutions for the second harmonic are shown in Fig. 4 for different values of  $\Delta s$ . Note that for  $\Delta s=0$ , one recovers the solution presented in Sec. 2.3. As  $\Delta s$  increases, the repetition period and the maximum conversion efficiency during SHG decrease.



**Figure 4** Growth of second harmonic amplitude with propagation for the phase mismatched case [Banerjee and Poon (1991)].

In what follows, we will determine the period for the case of large phase mismatch  $\Delta s \gg 1$  (Banerjee and Korpel, 1981). As we will show, the period is modified from the linear theory predictions due to the nonlinearity. By defining:

$$\sin^2 \alpha = \frac{v^2 - v_a^2}{v_b^2 - v_a^2} = \frac{v^2}{v_b^2} = v^2 v_c^2, \quad (2-36)$$

we obtain from Eq. (2-32):

$$\Pi_\zeta = \frac{2}{v_c} \int_0^{\pi/2} \frac{d\alpha}{\left[1 - \left(\frac{v_b}{v_c}\right)^2 \sin^2 \alpha\right]^{1/2}}. \quad (2-37)$$

Now,  $v_b^2$  can be simplified, using Eq. (2-34), for  $\Delta s \gg 1$  as:

$$v_b^2 \approx \left(\frac{2}{\Delta s}\right)^2 \left[1 - \frac{8}{(\Delta s)^2}\right]. \quad (2-38a)$$

Similarly,

$$v_c^2 \approx \left(\frac{\Delta s}{2}\right)^2 \left[1 + \frac{8}{(\Delta s)^2}\right]. \quad (2-38b)$$

Because  $\Delta s \gg 1$ ,  $\frac{v_b^2}{v_c^2} \ll 1$ , hence:

$$\Pi_\zeta \approx \frac{2}{v_c} \int_0^{\pi/2} d\alpha = \frac{\pi}{v_c};$$

hence:

$$\Pi_\zeta \approx \frac{2\pi}{\Delta s} \left[1 - \frac{4}{(\Delta s)^2}\right]. \quad (2-39)$$

The denormalized period can be calculated using Eq. (2-17) and shows that the period in  $z$  is modified by an amount proportional to the nonlinearity coefficient and the initial fundamental intensity, and inversely proportional to the linear phase mismatch  $\Delta k$ .

### 3 PHASE MATCHING IN ANISOTROPIC CRYSTAL

In Sec. 1, we had pointed out the physics of how in an  $oo-e$  type interaction, propagation of the fundamental at an angle to the  $c$ -axis can result in second

harmonic generation, which is phase-matched by using the crystal anisotropy. In a material with  $n_o > n_e$ , this is possible, as can be readily understood by considering the specific example of a SHG crystal-like KDP. For this material and  $\omega_0 = 2.715 \times 10^{15}$  rad/sec ( $\lambda_0 = 0.6943 \mu\text{m}$ ):

$$\begin{aligned} n_o^{\omega_0} &= 1.50502 & n_e^{\omega_0} &= 1.46532 \\ n_o^{2\omega_0} &= 1.53289 & n_e^{2\omega_0} &= 1.48711. \end{aligned} \quad (3-1)$$

Referring to Eq. (3-20) of Chap. 1, it is clear that:

$$n_e < n_e(\theta) < n_o. \quad (3-2)$$

Thus there will always exist an angle  $\theta_m$  along which:

$$n_e^{2\omega_0}(\theta_m) = n_o^{\omega_0} = 1.50502. \quad (3-3)$$

Hence:

$$n_o^{\omega_0} = \left[ \frac{\cos^2 \theta_m}{(n_o^{2\omega_0})^2} + \frac{\sin^2 \theta_m}{(n_e^{2\omega_0})^2} \right]^{-1/2}$$

or

$$\cos^2 \theta_m = \left[ \frac{(n_o^{\omega_0})^2 - (n_e^{2\omega_0})^2}{(n_o^{2\omega_0})^2 - (n_e^{2\omega_0})^2} \right] \frac{(n_o^{2\omega_0})^2}{(n_o^{\omega_0})^2} \quad (3-4)$$

which represents the angle, called the *critical phase matching angle*; the o-wave fundamental should meet with the optical axis for maximum efficiency (Ghatak and Thyagarajan, 1989).

As also noted from Chap. 1, the Poynting vector for the second harmonic will separate from the propagation direction of the fundamental and second harmonic due to crystal anisotropy. This phenomenon is referred to as *walkoff*. The typical value for the walkoff angle  $\rho$  is about  $2^\circ$  for KDP. In practice, the fundamental beam may be focused and have a transverse dimension  $w_0$ . Then the fundamental and second harmonic powers will separate after a distance  $L_w$  of approximately  $w_0/\tan \rho \approx w_0/\rho$ , leading to decrease in the conversion efficiency.

#### 4 NONLINEAR TRANSVERSE EFFECTS IN SECOND HARMONIC GENERATION

In all discussions above, we have concentrated on plane wave interactions between the fundamental and the second harmonic. However, in practice, the

fundamental beam is Gaussian, which may be often focused to increase the intensity and thus the second harmonic output. Furthermore, if the divergence of the input beam is about  $2^\circ$  or so (corresponding to a fundamental beam waist of the order of  $100\lambda$ ), the magnitude of the diffraction term becomes comparable to the walkoff, which may limit SHG efficiency. Accordingly, in what follows, we set up the problem mathematically, which can be used to analyze SHG when the input is a beam and in the presence of walkoff and phase mismatch, and present some numerical results for special cases. More general cases are analyzed in Pliszka and Banerjee (1993).

The model equations are a generalization of Eqs. (2-7a) and (2-7b) in the presence of walkoff and propagational diffraction. The effect of walkoff can be included by writing an additional derivative *w.r.t.* a transverse coordinate on the LHS of Eq. (2-7b). Diffraction, in the paraxial approximation, is handled by adding terms proportional to the transverse Laplacian on the RHS of Eqs. (2-7a) and (2-7b). Accordingly, the modified evolution equations become:

$$\frac{dE_e^{\omega_0}}{dz} = -j\sigma E_e^{\omega_0} E_e^{\omega_0} e^{-j\Delta kz} + \frac{j}{2k_1} \nabla_\perp^2 E_e^{\omega_0}, \quad (4-1)$$

$$\left( \frac{\partial}{\partial z} - \rho \frac{\partial}{\partial x} \right) E_e^{\omega_0} = -j\sigma E_e^{\omega_0 2} e^{j\Delta kz} + \frac{j}{2k_2} \nabla_\perp^2 E_e^{2\omega_0} \quad (4-2)$$

where

$$\sigma = \frac{\omega_0^2}{2c^2 k_1} \chi^{(2)}(-\omega_0; 2\omega_0, -\omega_0) = \frac{\omega_0^2}{c^2 k_2} \chi^{(2)}(-2\omega_0; \omega_0, \omega_0).$$

In the quasi-linear limit (negligible depletion of fundamental), diffraction plays a nonnegligible role, and it can be shown that (assuming  $\rho = 0$ ):

$$E_e^{2\omega_0}(x, y, z) = c(z) E_e^{\omega_0 2}(x, y, z) \quad (4-3)$$

where

$$c(z) = \frac{j\sigma}{E_e^{\omega_0}(0, 0, z)} \int_0^z E_e^{\omega_0}(0, 0, z') e^{j\Delta z'} dz'. \quad (4-4)$$

The above relation implies that the transverse profile of the second harmonic will also be Gaussian, with the width given by  $w_2(z) = (1/\sqrt{2})w_1(z)$ , where  $w_1(z)$  is the width of the fundamental Gaussian profile. Relation (4.3), when substituted into Eq. (4-1), implies that up to a first approximation, the propagation of the fundamental beam can be described as the so-called nonlinear Schrodinger (NLS) equation (see Chap. 4) with a complex  $z$ -dependent nonlinearity coefficient.

If the width of the fundamental beam is large ( $\approx 1000\lambda$ ), we are close to the diffraction-free limit. In this case, we can analytically consider the case when the fundamental amplitude is depleted. In this case:

$$I_2(x, y, z) = |E_e^{2\omega_0}(x, y, z)|^2 = I_1(x, y, 0)v^{-2}\text{sn}^2[(z/l_{\text{nl}})v, v^{-4}] \quad (4-5)$$

where  $\text{sn}$  is an elliptic function, and

$$\begin{aligned} v^{-1} &= \Delta k l_{\text{nl}}/4 + \left[1 + (\Delta k l_{\text{nl}}/4)^2\right]^{1/2}, \\ l_{\text{nl}} &= l_{\text{nl}}(x, y) = \frac{k_1}{\sigma |E_e^{\omega_0}(x, y, 0)|}, \\ l_{\text{nl}}(0, 0) &= L_{\text{nl}}; I_1(x, y, 0) = |E_e^{\omega_0}(x, y, 0)|^2, \end{aligned} \quad (4-6)$$

with the special hyperbolic tangent-type solution corresponding to the separation in the  $\Delta k = 0$  ( $v = 1$ ) case.

For the case  $\Delta k = 0$ , Fig. 5 shows the evolution of the fundamental profile. Note that the intensity depletion is faster in the center of the beam than in the wings, as  $l_{\text{nl}}(x, y) > L_{\text{nl}}$ , resulting in an initial flattening of the radial intensity profile of the fundamental and a subsequent dip in the center, corresponding to a ring structure in the output plane. The corresponding profiles of the second harmonic are approximately Gaussian.

In the  $\Delta k \neq 0$  case, the period of the elliptical functions is given by

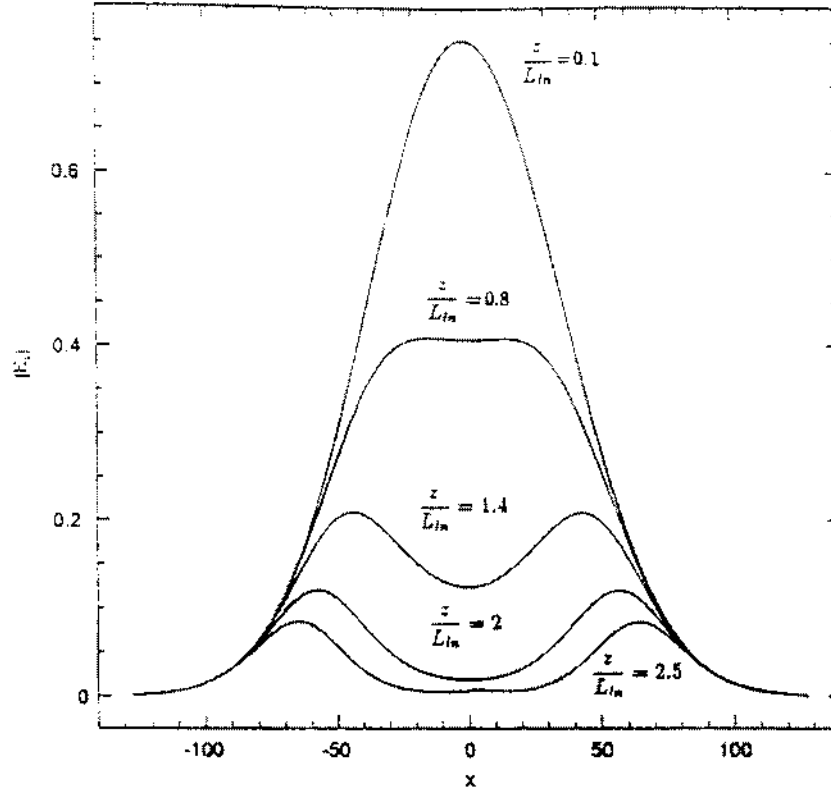
$$A = \left[ \frac{\Delta k}{4} + \left( \frac{1}{l_{\text{nl}}^2} + \frac{(\Delta k)^2}{16} \right)^{1/2} \right]^{-1}. \quad (4-7)$$

Thus, one observes a slightly shorter period of the oscillations of the intensity transfer in the center of the beam and slightly longer at the tails, which distorts the Gaussian beam profile (see Fig. 6a). As a result, the total power of each wave is not a periodic function of the propagation distance but tends to a constant value in an oscillatory fashion, as shown in Fig. 6b.

For the case of diffraction, phase mismatch, and walkoff, the reader is referred to Pliszka and Banerjee (1993) for details. Suffice to state that the second harmonic profile develops secondary peaks due to *walkoff*, and the primary peak is displaced *w.r.t.* the fundamental's Gaussian peak.

## 5 SHG IN A MEDIUM WITH SECOND-ORDER AND THIRD-ORDER NONLINEAR SUSCEPTIBILITIES

In general, all materials have higher-order nonlinearities. With higher laser powers, the role of third-order nonlinearities becomes important during SHG, as stated in Sec. 1. In what follows, we will once again formulate the

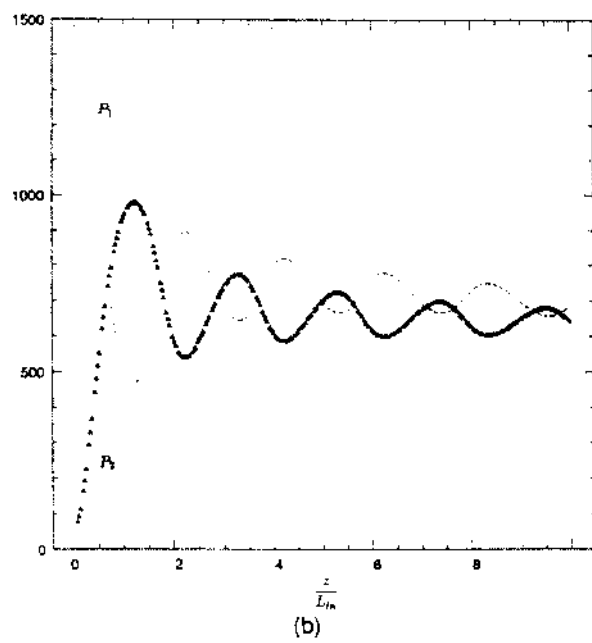
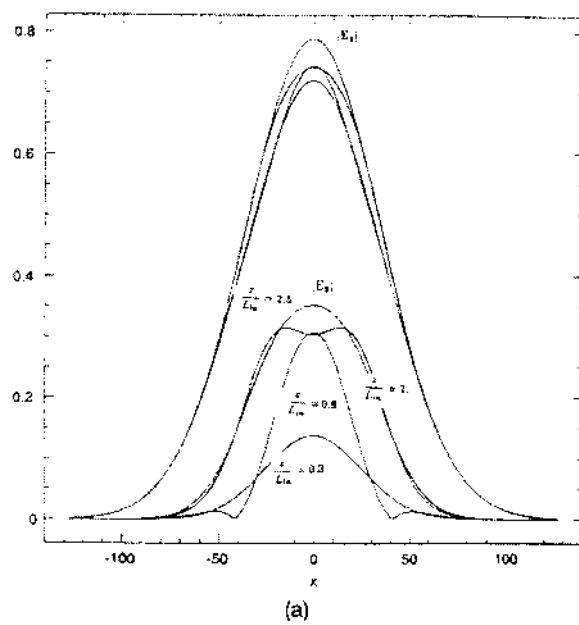


**Figure 5** Transverse profile for the fundamental amplitude for negligible diffraction and no linear phase mismatch [Pliszka and Banerjee (1993)].

problem mathematically and qualitatively discuss some solutions. Readers interested in details can refer to Harter and Brown (1982), Razumikhina et al. (1984), and Choe et al. (1991).

In the presence of both quadratic and cubic nonlinearities, Eq. (2-1) has a second term,  $-(1/\varepsilon)\partial^2 \mathbf{P}^{(3)}/\partial t^2$  on the RHS. If the same procedure as that used to derive Eqs. (2.19a), (2.19b), and (2.19c) is employed, it can be shown

**Figure 6** (a) Transverse profile of the fundamental and second harmonic for the case of negligible diffraction and  $\Delta k L_{nl} = 4$  for various propagation distances  $z/L_{nl}$ . (b) Powers  $P_{1,2}$  of fundamental and second harmonic, respectively, for negligible diffraction and  $\Delta k L_{nl} = 0.5$  [Pliszka and Banerjee (1993)].



that the evolution equations for  $u$  and  $v$  remain unchanged, whereas the equation describing the relative phase takes the form:

$$\frac{d\phi}{d\zeta} = \Delta s - \left(2v - \frac{u^2}{v}\right) \cos\phi + T_1 u^2 + T_2 v^2, \quad (5-1)$$

where  $T_{1,2}$  is proportional to the third-order nonlinear susceptibilities. Once again, as before,  $u^2 + v^2 = 1$ . This also implies that the third-order nonlinearity plays the role of modulating the phase and that, in some cases, it may be partially possible to balance the linear phase mismatch by this nonlinearly induced phase, resulting in greater SHG conversion efficiency.

As in the previous analysis, the solution of Eqs. (2-19a) and (2-19b), and (5-1) can be expressed in terms of elliptical integrals similar to Eq. (2-31):

$$\zeta = \pm \frac{1}{2} \int_h^\infty \frac{\alpha h}{\sqrt{D(h)}} \quad (5-2)$$

where:

$$\begin{aligned} h &= 1/v^2, \\ D(h) &= h^3 - (T_4^2 + 2)h^2 + (1 - 2T_3T_4)h - T_3^2, \\ T_3 &= (T_2 - T_1)/4; (\Delta s + T_1)/2. \end{aligned} \quad (5-3)$$

Periodic and aperiodic solutions of Eq. (5-2) exist. Periodic solutions may be either cn, sn functions, or trigonometric, whereas aperiodical solutions may be hyperbolic or algebraic functions. In principle, 100% conversion efficiency for a particular crystal is possible for a particular crystal length and initial fundamental power (for details, see Choe et al., 1991); in practice, absorption and the finite width of the beam may determine the maximum efficiency.

## 6 SUMMARY

In this chapter, we have studied SHG in a material with quadratic nonlinearity, and later, with quadratic and cubic nonlinearities. We have examined the effect(s) of phase mismatch, beam size, etc., on the second harmonic efficiency. Effects of walkoff have been mentioned, and a way to achieve quasi-phase matching in a crystal has been discussed. We have also shown that a quadratic nonlinearity with phase mismatch can “simulate” a cubic nonlinearity, which can support solitons. More details on solitons appear in Chap. 8. Also, more on phase matching during SHG is discussed in the context of periodic poling of the material and photonic bandgap structures in Chap. 12.

## 7 PROBLEMS

1. Develop the evolution equations for the fundamental and second harmonic with and without linear phase mismatch using nonlinear evolution Eq. (3-4) and assuming only a quadratic nonlinearity. Compare your results with the equations derived using the more rigorous approach.
2. Repeat Problem 1 assuming quadratic and cubic nonlinearities. Again, compare your results with the more rigorous approach.
3. Repeat Problem 1 assuming only a cubic nonlinearity but now assuming a DC electrical field. Find the “effective” second-order nonlinearity coefficient.
4. Investigate subharmonic generation starting from a finite amount of power in the fundamental  $v(0)$  and a small amount of power in the subharmonic (starting from noise), which can be taken as  $\delta v(0)$ .
5. Determine the quasi-matching angle in ADP for a fundamental frequency of  $1 \mu\text{m}$  and for which

$$\begin{aligned} n_o^{\omega_0} &= 1.495628, & n_e^{\omega_0} &= 1.460590 \\ n_o^{2\omega_0} &= 1.514498, & n_e^{2\omega_0} &= 1.472068. \end{aligned}$$

6. Develop the evolution equations for three interacting frequencies  $\omega_1, \omega_2 \neq 2\omega_1$ , and  $\omega_3 = \omega_1 + \omega_2$  in a material having a general second-order susceptibility tensor and a relative phase mismatch due to  $\Delta k = k_3 - k_2 - k_1$ , where the  $k$  values are propagation constants corresponding to the respective frequencies. Attempt to find the solution to the system of equations by assuming that only  $\omega_{01,02}$  initially exists.
7. Show that if the direction of propagation deviates from the phase matching angle, the conversion efficiency decreases and follows, approximately, as  $\sin^2 \beta / \beta^2$ , where  $\beta$  denotes the departure from the phase matching angle.

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# 4

## Self-Refractive of Optical Beams

In the previous chapter, we have discussed the effect of a second-order nonlinearity on wave propagation, with emphasis on second-harmonic generation using plane waves, and later, beams. In this chapter, we investigate in more detail the effect(s) of optical nonlinearities on beam propagation. In particular, we will first investigate the effect of a cubic nonlinearity because this gives rise to *self-phase modulation* (SPM), which can modify the beam profile during its travel. Later we will also show in more depth how the presence of only a quadratic nonlinearity along with phase mismatch can simulate the effect of a cubic nonlinearity.

To be more specific, we will first show that the contribution to wave propagation from the nonlinear third-order susceptibility  $\chi^{(3)}$  can, in simple cases, be equivalently described in terms of an effective *nonlinear refractive index coefficient*  $n_2$ , to be made more precise later. We will then investigate the propagation of Gaussian beams through such a nonlinear medium and show that it can *self-focus* or *self-defocus* depending on the sign of  $n_2$ . To do this analysis, we will derive a simple differential equation that describes the variation of the width of the Gaussian beam during its travel through a finite sample of such a nonlinear medium. It turns out that in many practical applications, the sample is “thin,” meaning that the beam does not appreciably diffract, in the absence of nonlinearities, while traversing the sample. In this case, the exiting beam only undergoes a nonlinearly induced phase change, and the far-field profile of the beam is easy to investigate. In fact, such a technique is often used to determine the sign and magnitude of the nonlinearity, a procedure often referred to as the *z-scan* technique. Furthermore, it is also easy to show that an asymmetric beam, upon passage through such a nonlinear slice, undergoes *self-bending* in the far field. Finally, in this chapter, we will evaluate an effective  $n_2$  or  $\chi^{(3)}$  due to the presence of only  $\chi^{(2)}$  and a phase mismatch during SHG. Except in this particular subsection, no second harmonic will be assumed to be present in any other discussion because,

as shown in Chap. 2, SHG with  $\chi^{(3)}$  is only possible in the presence of an externally applied d.c. field.

## 1 THE NONLINEAR REFRACTIVE INDEX COEFFICIENT $n_2$

In Chap. 2, it was shown that the third-order polarization  $P_p^{(3)\omega_0}$  can be expressed as

$$P_p^{(3)\omega_0} = (3\epsilon_0/4)\chi^{(3)}(-\omega_0; \omega_0, \omega_0, -\omega_0): E_p^{\omega_0} E_p^{\omega_0} E_p^{\omega_0*} \quad (1-1a)$$

or, equivalently,

$$P_{pi}^{(3)\omega_0} = (3\epsilon_0/4)\chi_{ijkl}^{(3)}(-\omega_0; \omega_0, \omega_0, -\omega_0): E_{pj}^{\omega_0} E_{pk}^{\omega_0} E_{pl}^{\omega_0*}. \quad (1-1b)$$

Consider a simple case where the optical fields are along the 1 or  $x$  direction and  $\chi_{1111}^{(3)}$  is the dominant nonzero  $\chi^{(3)}$  tensor element; then

$$P_{p1}^{(3)\omega_0} = (3\epsilon_0/4)\chi_{1111}^{(3)}(-\omega_0; \omega_0, \omega_0, -\omega_0): E_{p1}^{\omega_0} E_{p1}^{\omega_0} E_{p1}^{\omega_0*}. \quad (1-2)$$

Let us now examine Eq. (3-40) in Chap. 2. If we take

$$E = (1/2)E_p^{\omega_0} e^{j\omega_0 t} \mathbf{a}_x + \text{c.c.}; \quad P^{\text{NL}} = (1/2)P_p^{(3)\omega_0} e^{j\omega_0 t} \mathbf{a}_x + \text{c.c.} \quad (1-3)$$

and substitute in Eq. (3-40) of Chap. 2, and assume the phasors to be only functions of  $z$  for simplicity, then

$$d^2 E_p^{\omega_0} / dz^2 + k_0^2 n^2 E_p^{\omega_0} = 0 \quad (1-4)$$

where the subscript 1 has been dropped for  $E_p^{\omega_0}$  since it signifies the  $x$  component. In Eq. (1-4),

$$n = n_0 + n_2 |E_p^{\omega_0}|^2 \quad (1-5)$$

with

$$n_2 = (3/8n_0)\chi_{1111}^{(3)}(-\omega_0; \omega_0, \omega_0, -\omega_0). \quad (1-6)$$

The parameter  $n_2$  is termed the *nonlinear refractive index coefficient* and is proportional to  $\chi^{(3)}$ . Note that using the alternative approach described in Sec. 3.5 of Chap. 2, one can find a relation between  $\beta_3$  and  $\chi^{(3)}$  as well, and

hence, between  $\beta_3$  and  $n_2$ . This will not be derived in rigor, but is written below for completeness:

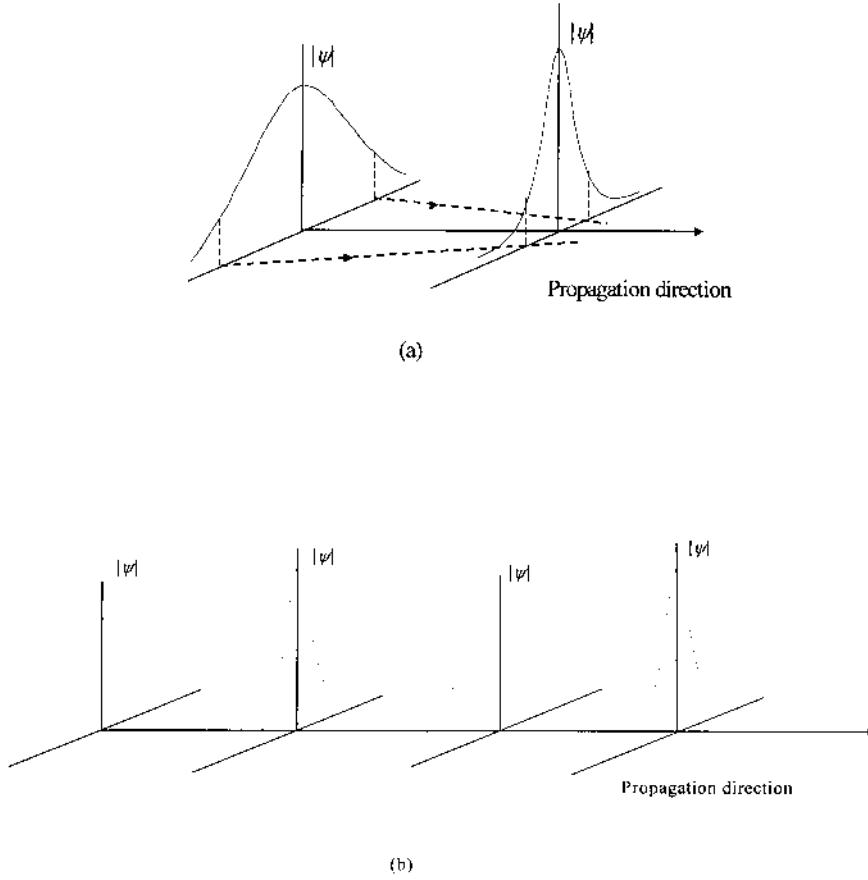
$$\beta_3 = -(3/2n_0^2)\chi^{(3)}. \quad (1-7)$$

The nonlinear refractive index coefficient  $n_2$  (and, equivalently,  $\beta_3$ ) is sometimes a convenient way to analyze the physics of *self-refraction* without getting involved in the mathematical manipulation of tensors. In an actual practical situation involving a nonlinear crystal, particular attention should be paid to which components of  $\chi^{(3)}$  play the role in self-refraction. In what follows, we will analyze self-focusing and self-defocusing, as well as self-bending of optical beams using  $n_2$  as the nonlinearity parameter.

## 2 SELF-REFRACTION OF GAUSSIAN BEAMS

As one of the effects of a cubic nonlinearity on wave propagation, we will first consider *self-refraction* of a beam. To get a physical picture, consider a beam with a Gaussian profile, as shown in Fig. 1(a). If the medium is cubically nonlinear, we can describe the nonlinearity in terms of  $n_2$ , the nonlinear coefficient of the refractive index, which is related to  $\beta_3$  as given in Eqs. (1-6) and (1-7). If  $n_2 > 0$ , a region transverse to the propagation direction with greater amplitude experiences a greater refractive index than a region with lower amplitude. The result is an intensity-dependent refractive index profile like that in graded-index optical fibers. If we trace rays, these would appear to bend towards the axis of propagation, indicating a reduction of the beam waist size and hence an increase in the on-axis amplitude. This simple picture would suggest that the on-axis amplitude should tend to infinity; however, this does not occur because diffraction puts a limit to the minimum waist size. This is shown in Fig. 1(b). Heuristically speaking, this makes sense because the amount of diffraction (as predictable from the angle of diffraction) depends on the ratio of the wavelength to the waist size. For an arbitrary initial beam profile, therefore, we would expect initial reduction of beam waist size before diffraction effects start to dominate and spread the beam again, resulting in *periodic focusing*. It turns out that although this is mostly true for a beam in a two-dimensional geometry (meaning one transverse dimension), it may not be true for the three-dimensional case. It is also possible to find the right beam profile for which the beam-spreading effect of diffraction balances the focusing effect of the cubic nonlinearity (Chiao et al., 1964; Akhmanov et al., 1966).

For the opposite kind of nonlinearity ( $n_2 < 0$ ), it is easy to argue that the beam will spread more than it does for the linear diffraction-limited case (see Fig. 2).

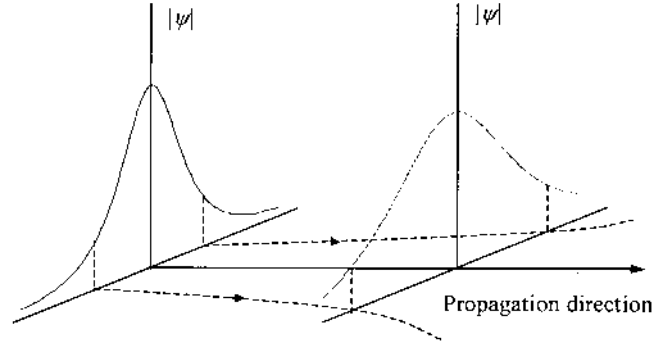


**Figure 1** Self-focusing of a Gaussian beam. (a) Ray trajectory and (b) periodic self-focussing.

In what follows, we will first derive an equation that describes the propagation of a Gaussian beam through a finite sample of a cubically nonlinear material. For simplicity, we will treat the optical field as a scalar and designate the real amplitude as  $\psi$ , with  $\psi_p$  and  $\psi_e$  denoting the phasor and the envelope, respectively.

Before beginning, let us reiterate two results for the  $q$  parameter of Gaussian beams, one of which was discussed in Chap. 1. First, recall that when a Gaussian beam travels a distance  $z$ , the  $q$  parameter changes according to

$$q' = q + z \quad (2-1a)$$



**Figure 2** Self-defocusing of a beam. The beam spreads more than in the linear diffraction-limited case.

where  $q'$ ,  $q$  are the new and original  $qs$  of the beam. Secondly, when a Gaussian beam passes through a lens, the  $q$  parameter change can be written as

$$1/q' = 1/q - 1/f \quad (2-1b)$$

where  $f$  is the focal length of the lens (Poon and Banerjee, 2001). We will use these two simple rules of  $q$  transformation to track the propagation of a Gaussian beam through a cubically nonlinear medium. This is because the  $n_2$  in turn is responsible for inducing a lenslike effect on the Gaussian beam, as shown below.

Consider propagation by a distance  $\Delta z$ . The overall effect on the  $q$  is a combination of propagational diffraction and induced lensing. The former changes the  $q$  by  $z$ . To assess the effect of the latter, note, using Eq. (2-1b),

$$\Delta q = q^2/f_{\text{ind}} \quad (2-2)$$

where  $f_{\text{ind}}$  is the nonlinearly *induced focal length* of a slice  $\Delta z$  of the nonlinear medium when a Gaussian beam propagates in it. The overall change due to both the effects is then

$$\Delta q = \Delta z + q^2/f_{\text{ind}}(\Delta z). \quad (2-3)$$

To determine  $f_{\text{ind}}(\Delta z)$ , assume a Gaussian of the form

$$|\psi_e(x, y, z)| = a(z) \exp - (x^2 + y^2)/w^2(z). \quad (2-4)$$

Since

$$n = n_0 + n_2 |\psi_e|^2 \approx n_0 - 2n_2 a^2(z)(x^2 + y^2)/w^2(z),$$

the nonlinearly induced phase change is

$$\Delta\phi = k_0 n \Delta z = k_0 [n_0 - 2n_2 a^2(z)(x^2 + y^2)w^2(z)] \Delta z$$

so that

$$f_{\text{ind}}(z) = n_0 w^2(z) / 4n_2 a^2(z) \Delta z. \quad (2-5)$$

Note that  $f_{\text{ind}}(z)$  is inversely proportional to  $n_2 a^2(z)$  and  $\Delta z$ . The longer the sample length, the more the nonlinear induced phase, implying a stronger lensing (smaller  $f_{\text{ind}}$ ). Also, the stronger the nonlinearity parameter  $n_2$  and/or on-axis intensity  $a^2(z)$ , the stronger the focusing because the change in  $n$  is larger. Substituting Eq. (2-5) into Eq. (2-3) and taking the limit as  $\Delta z \rightarrow 0$ , we obtain the differential equation

$$dq/dz = 1 + \frac{4n_2 a^2(z) q^2}{n_0 w^2(z)}. \quad (2-6)$$

Noting that

$$a^2(z) w^2(z) = (2/\pi) \eta P \quad (2-7)$$

where  $P$  is the beam power and  $\eta$  is the characteristic impedance (see Eq. (2-29) in Chap. 1) Eq. (2-7) can be rewritten as

$$dq/dz = 1 + \frac{8n_2 \eta P q^2}{\pi n_0 w^4(z)}. \quad (2-8)$$

Equation (2-8) is thus the required differential equation we will use to study self-diffraction. However, note that  $q$  itself is a function of  $w$  and the radius of curvature  $R$  of the Gaussian beam. This relationship is given by (Poon and Banerjee, 2001)

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j \frac{\lambda_0}{\pi n_0 w^2(z)}. \quad (2-9)$$

We will thus use Eq. (2-9) to resolve Eq. (2-8) into two equations, in  $R$  and  $w$ . Thereafter, we will eliminate one of the variables to yield a nonlinear second-order ordinary differential equation (ODE) in  $w(z)$  which will be solved to analyze self-refraction.

Upon differentiating Eq. (2-8) and after some algebra, using Eq. (2-9), we get the imaginary and real parts as

$$\frac{1}{R(z)} = \frac{1}{w} \frac{dw}{dz} \quad (2-10)$$

and

$$\frac{1}{R^2} \frac{dR}{dz} = \frac{n_0^2 \pi^2 w^4 - \lambda_0^2 R^2}{(\pi n_0 w^2 R)^2} = \frac{8n_2 \eta P}{\pi n_0 w^4}. \quad (2-11)$$

Finally, using Eq. (2-10) in Eq. (2-11) and further algebra, we get

$$\frac{d^2 w}{dz^2} = \frac{\lambda_0^2/n_0^2\pi^2 - 8n_2\eta P/\pi n_0}{w^3}, \quad (2-12)$$

which is the required equation.

An examination of Eq. (2-12) shows that the first term on the right-hand side shows the effect of propagational diffraction, whereas the second term depicts the nonlinear contribution. To reassure ourselves of the correctness of Eq. (2-12), we will first take  $n_2 = 0$  and derive earlier expressions for the beam width during Fresnel diffraction, as derived in Chap. 1. To do this, observe that for this case

$$\frac{d^2 w}{dz^2} = \frac{\lambda_0^2/n_0^2\pi^2}{w^3}. \quad (2-13)$$

Multiplying both sides by  $dw/dz$ , integrating and setting the initial condition,  $dw/dz = 0$  at  $z = 0$  and  $w = w_0$  at  $z = 0$ , we get

$$\left(\frac{dw}{dz}\right)^2 = \frac{\lambda_0^2}{n_0^2\pi^2} \left(\frac{1}{w_0^2} - \frac{1}{w^2}\right). \quad (2-14)$$

The solution of Eq. (2-14) can be operationally written as

$$\frac{\lambda_0}{n_0\pi} z = \frac{w_0}{2} \int_{w_0^2}^{w^2} \frac{dw^2}{\sqrt{w^2 - w_0^2}}. \quad (2-15)$$

After some algebra, Eq. (2-15) can be integrated and expressed as

$$w^2(z) = w_0^2 \left(1 + \left(\frac{z}{z_R}\right)^2\right), \quad (2-16)$$

in agreement with Eq. (4-19) in Chap. 1.

For the nonlinear case  $n_2 \neq 0$ , the solution of Eq. (2-12) can be found in a similar way. Before getting to the general solution, however, we would like to make one observation: nonlinearity exactly balances the effect of diffraction when

$$\lambda_0^2/n_0^2\pi^2 = 8n_2\eta P/\pi n_0$$

implying that for

$$P = P_{\text{cr}} = \lambda_0^2/8\pi n_0 n_2 \eta, \quad (2-17)$$

the Gaussian beam should remain unchanged during propagation. As we shall see shortly, the beam profile that propagates without distortion is not

exactly Gaussian, but, nonetheless, has the same general shape. For powers greater than the *critical power*  $P_{\text{cr}}$  and for  $n_2 > 0$ , the Gaussian beam will *self-focus* [Ghatak and Thyagarajan (1989)]. We remark, however, in reality, the balance between nonlinearity and diffraction slightly changes the profile of the beam to non-Gaussian, and hence a nonspreading solution is not really a Gaussian but has a *sech* profile. This will be more rigorously derived below.

Returning to the general case of Gaussian propagation, note that for  $n_2 > 0$ , Eq. (2-12) can be rewritten as

$$\frac{d^2 w}{dz^2} = \frac{8n_2\eta(P - P_{\text{cr}})/\pi n_0}{w^3}. \quad (2-18)$$

Following steps similar to that done for the  $n_2 = 0$  case, we obtain

$$w^2(z) = w_0^2 - 8n_2\eta(P - P_{\text{cr}})z^2/\pi n_0. \quad (2-19)$$

Hence, at

$$z = z_f = \sqrt{\frac{\pi n_0 w_0^4}{8n_2\eta(P - P_{\text{cr}})}} \quad (2-20)$$

the Gaussian beam is expected to self-focus. In reality, the width never goes to zero, but arrives at a minimum before the diffraction effect dominates and the beam spreads again. This gives rise to periodic focusing through the self-guiding action of the Gaussian beam.

For  $n_2 < 0$ , the solution can be either exactly found or derived by substituting the exact expression for  $P_{\text{cr}}$  from Eq. (2-17) into Eq. (2-19), and noting that  $P_{\text{cr}} < 0$  since  $n_2 < 0$ .

As pointed out earlier, our discussion thus far was limited to Gaussian beams. To find the exact profile of nonspreading solutions, we need to derive the nonlinear equation describing beam propagation in cubically nonlinear media. This equation has the same form as the paraxial wave equation [Eq. (4-5) in Chap. 1] but modified to include the effect of nonlinearity.

For a variation, we will derive the required equation starting from Eq. (3-39) of Chap. 2 with  $\beta_3 \neq 0$  and  $\beta_2 = 0$ . Since we are considering *beams*, we put

$$\psi(x, y, z) = \text{Re} [\psi_e(x, y, z) \exp j(\omega_0 t - k_0 z)] \quad (2-21)$$

and substitute into Eq. (4-9) of Chap. 2 with  $\beta_2 = 0$ . After straightforward algebra, we get

$$2jk_0 \partial \psi_e / \partial z = \nabla_{\text{t}}^2 \psi_e - (\beta_3 k_0^2 / 2) |\psi_e|^2 \psi_e, \quad (2-22)$$

where we have only retained contributions around  $(\omega_0, k_0)$ . Furthermore, we have assumed  $\omega_0/k_0 = v$  and  $\psi_e$  to be a slowly varying function of  $z$  in the sense that

$$\left| \frac{\partial^2 \psi_e}{\partial z^2} \right| \ll k_0 \left| \frac{\partial \psi_e}{\partial z} \right|. \quad (2-23)$$

In Eq. (2-22),  $\nabla_t^2$  denotes the transverse Laplacian. We remark that Eq. (2-22) with  $\beta_3 = 0$  is identical to Eq. (4-5) in Chap. 1, which describes diffraction of a beam in a linear medium. Thus, we can regard Eq. (2-22) to be the nonlinear extension of Eq. (4-5) in Chap. 1. In Eq. (2-22), the first term on the right-hand side is due to diffraction, whereas the second term represents the nonlinear contribution. The equation has the same form as the *nonlinear Schrödinger (NLS) equation* (Korpel and Banerjee, 1984; Agrawal, 1989), which is used to explain *soliton* propagation through fibers (Mollenauer and Stolen, 1982). We will discuss more about soliton propagation later on in this book. Note that we cannot solve Eq. (2-22) using the Fourier transform techniques that were used to solve Eq. (4-5) in Chap. 1 because Eq. (2-22) is a nonlinear partial differential equation (PDE).

In our quest for the expression of  $|\psi_e|$  that does not depend on  $z$ , we substitute (Banerjee and Poon, 1991)

$$\psi_e(x, y, z) = a(x, y) \exp -jkz \quad (2-24)$$

into Eq. (2-22) to get

$$\nabla_t^2 a = -2\kappa k_0 a + (\beta_3 k_0^2 / 2) a^3. \quad (2-25)$$

Consider first the case where we have one transverse direction, namely,  $x$ . Equation (2-25) then reads

$$d^2 a / dx^2 = -2\kappa k_0 a + (\beta_3 k_0^2 / 2) a^3. \quad (2-26)$$

Multiplying both sides by  $2da/dx$  and integrating, we get

$$(da/dx)^2 = -2\kappa k_0 a^2 + (\beta_3 k_0^2 / 4) a^4, \quad (2-27)$$

where we have neglected the integration constant. We can recast Eq. (2-27) in the form

$$x = \int \frac{da}{\sqrt{-2\kappa k_0 a^2 + (\beta_3 k_0^2 / 4) a^4}}. \quad (2-28)$$

Equation (2-28) is in the form of an elliptic integral, which we first encountered while analyzing second-harmonic generation. The solution is in the form

$$a(x) = A \operatorname{sech} Kx, \quad (2-29)$$

where

$$A = (8\kappa/\beta_3 k_0)^{1/2}, \quad K = 1/(-2\kappa k_0)^{1/2}. \quad (2-30)$$

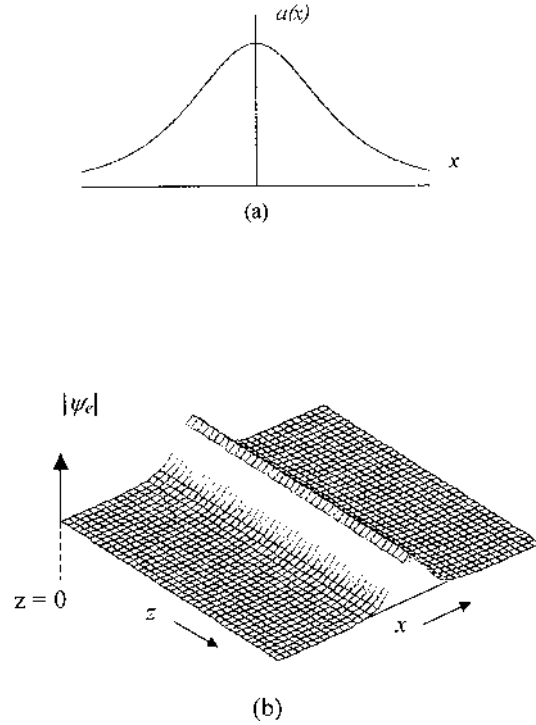
We note from above that  $\kappa < 0$  and  $\beta_3 < 0$  for a physical solution. Now  $\beta_3 < 0$  implies  $n_2 > 0$  [see Eqs. (1-6) and (1-7)], in agreement with our heuristic description for self-focusing. A plot of the beam profile, using the split-step beam propagation method (see Appendix A) is shown in Fig. 3.

For two transverse directions, namely,  $x$  and  $y$ , we only consider the case where there is radial symmetry and express the transverse Laplacian in polar coordinates as

$$\partial^2/\partial x^2 + \partial^2/\partial y^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r. \quad (2-31)$$

Using the definitions

$$a = (2\kappa/\beta_3 k_0)^{1/2} \tilde{a}, \quad r = (-1/2\kappa k_0)^{1/2} \tilde{r}, \quad (2-32)$$



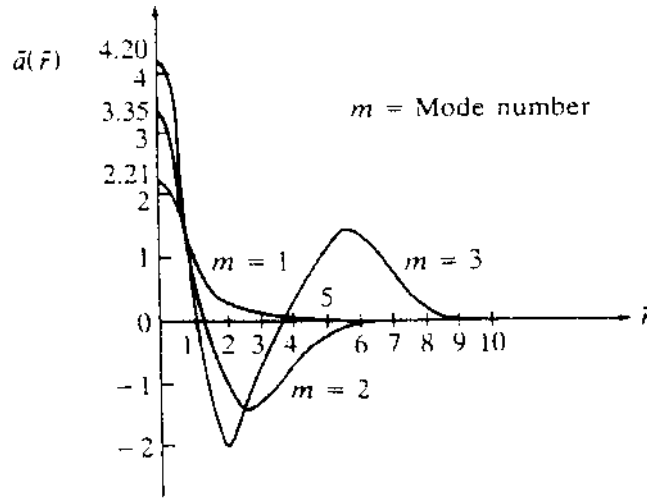
**Figure 3** (a) Stationary nonspreading sech-type solution in one transverse dimension. (b) Propagation of the nonspreading solution.

we rewrite Eq. (2-25) in the form

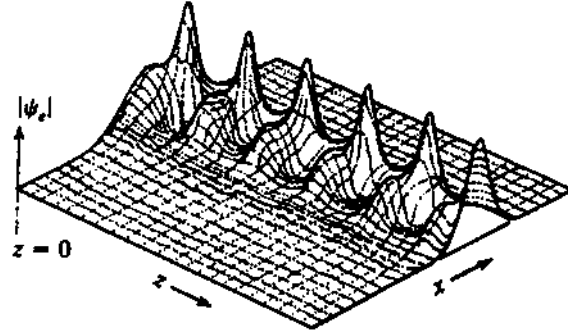
$$\frac{d^2 \tilde{a}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{a}}{d\tilde{r}} - \tilde{a} + \tilde{a}^3 = 0. \quad (2-33)$$

This ODE has no analytic solutions; the solutions that tend to zero as  $\tilde{r} \rightarrow \infty$  are obtained by numerical methods Chiao et al., 1964; Haus, 1966). These are shown in Fig. 4, and are extremely sensitive to the initial condition  $\tilde{a}(0)$ . In a way, these solutions are reminiscent of the modes in a *graded-index fiber*. Note that the solutions can be regarded as being *multimodal* in nature, with the mode number  $m$  depending on the initial condition  $\tilde{a}(0)$ .

The solutions discussed above are *nonspreading* or *diffraction-free* beam profiles in a cubically nonlinear medium. The general solution to an arbitrary initial beam profile is difficult to obtain, since the PDEs are nonlinear. We briefly comment that an approximate solution for such an arbitrary initial beam profile can be found by starting from Eq. (2-22), writing down the *nonlinear eikonal equations* by resolving  $\psi_e$  into its real amplitude and phase, and assuming the nature of solutions for these quantities (Banerjee et al., 1983). The calculations confirm periodic behavior in two transverse dimensions, for  $n_2 > 0$ . The detailed analysis is outside the scope of our discussion. We end this section by showing numerically computed pictures (using the beam propagation method, Appendix A) of the periodic behavior of an



**Figure 4** Numerically obtained higher dimensional soliton solutions showing amplitude profiles for higher order modes [Poon and Banerjee (2001)].

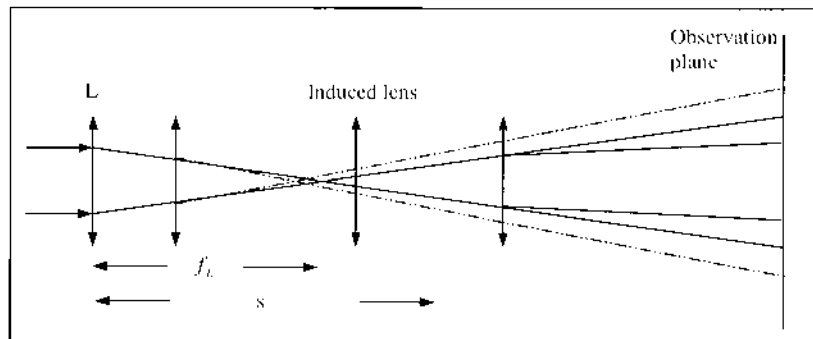


**Figure 5** Numerical solutions to the NLS equation showing amplitude profiles as a function of propagation for initial Gaussian profile in one transverse dimension. [Korpel et al. (1986)].

initial Gaussian profile (see Fig. 5) in one transverse dimension (Korpel et al., 1986), whose amplitude and width are close to the values in Eq. (2-30).

### 3 FOCUSED GAUSSIAN BEAM IN A KERR SLICE: THE z-SCAN METHOD

In this section, we present the results of propagation of a (circular) Gaussian beam that is initially focused by an external lens and is then incident on a slice of a cubically nonlinear Kerr medium. This case is important because as explained below, it provides a basis for the measurement of the nonlinear refractive index coefficient  $n_2$ . Fig. 6 shows a simple ray diagram to demon-



**Figure 6** Geometrical optics analysis of z-scan measurement for nonlinear refractive index.

strate what happens if the cubically nonlinear material is thin. The propagating Gaussian beam induces its own lens, which is positive for a material having a positive  $n_2$ , for reasons discussed in the previous section. Note that if the position of the induced lens is at the back focal plane of the external lens, the former does not affect the propagation of the beam behind the nonlinear material. The situation is thus the same as the propagation of the externally focused Gaussian beam in a linear diffraction-limited environment. However, if the separation between the external lens and the sample is less than the focal length  $f$  of the external lens, the beam in the far field is wider than the linear diffraction-limited case. Conversely, if the lens-sample separation is larger than  $f$ , the beam width is smaller. This is the basis of the *z-scan* measurement (Sheik-bahae et al., 1989; Banerjee et al., 1991): For a material with a positive  $n_2$ , the scanning of the thin nonlinear sample through the back focal plane of the external lens starting from a position to the left of the focus yields far-field profiles that are initially larger and eventually smaller than the width in the linear diffraction-limited case. The opposite is true for materials with a negative  $n_2$ .

The simpler ray picture described above can be augmented by using a  $q$  parameter approach to Gaussian-beam propagation, as discussed above. Recall that the  $q$  of a Gaussian beam in air is defined through the relation  $1/q = 1/R - j(2/k_0 w^2)$ , where  $R$  and  $w$  denote the radius of curvature and the beam width, respectively, and  $k_0$  is the propagation constant in air. Assume a Gaussian beam with a waist  $w_0$  (implying initially plane wave fronts) incident on the external lens. The Gaussian can be describe by a  $q = q_0 = jz_{R0}$ , where  $z_{R0} = k_0 w_0^2/2$  is called the *Rayleigh length* of the incident Gaussian. The  $q = q_{\text{obs}}$  of the Gaussian at a distance  $d$  from the nonlinearly induced lens of the focal length  $f_{\text{ind}}$ , located at a distance  $s = f + \Delta s$  behind the external lens is given by (Banerjee and Misra, 1991)

$$q_{\text{obs}} = \frac{f_{\text{ind}}[q_0 f + (f - q_0)(f + \Delta s)]}{f_{\text{ind}}(f - q_0) - [q_0 f + (f - q_0)s]} + d. \quad (3-1)$$

In deriving Eq. (3-1), we have used the laws of  $q$  transformation on the initial Gaussian beam passing through the external lens, being translated a distance  $f + \Delta s$ , passing through the induced lens, and finally being translated a distance  $d$  to the observation plane. We must point out that the focal length  $f_{\text{ind}}$  of the induced lens is also a function of  $\Delta s$ , because it depends on the beam intensity at the location of the nonlinear sample. In fact, the focal length  $f_{\text{ind}}$  of the nonlinearly induced lens can be expressed in terms of the beam power  $P$ , the beam width  $w$  at the position  $\Delta s$ ,  $f$ , the linear refractive index  $n_0$ , the nonlinear refractive index coefficient  $n_2$  defined through Eq. (1-5) where  $E_p^{(0)} = E$  is the optical field in the nonlinear material of length  $L$ .

As shown in Sec. 2, the focal length of the induced lens can then be written as

$$f_{\text{ind}}(\Delta s) \frac{n_0 w^2(\Delta s)}{4n_2 |E|^2 L} \approx \frac{\pi f^4 n_0}{k_0^4 w_0^4 \eta P L n_2} \quad (3-2)$$

for small  $\Delta s$ , where  $\eta$  is the (linear) characteristic impedance of the nonlinear medium. Note that in Eq. (3-2), the beam width  $w$  is given by

$$w^2(\Delta s) = w_f^2 \left[ 1 + \frac{4(\Delta s)^2}{k_0 w_f^4} \right] \quad w_f \equiv \frac{2f}{k_0 w_0}. \quad (3-3)$$

In Eq. (3-3),  $w_f$  denotes the waist size of the Gaussian beam at the back focal plane of the external lens. In deriving Eq. (3-2), one has to use the power conservation relation  $|E|^2 w^2 = (4/\pi)\eta P$  to reexpress the dependence of the induced focal length on the field amplitude in terms of the corresponding width at the same point.

Our final interest is to find the imaginary part of  $1/q_{\text{obs}} (= -2/k_0 w_{\text{obs}}^2)$ , in order to ascertain the width  $w_{\text{obs}}$  of the Gaussian beam on the observation plane. Simple algebra leads us the relation

$$\begin{aligned} & \frac{k_0 w_{\text{obs}}^2}{2} \\ &= \frac{[(ff_{\text{ind}}d - f^2d + f^2f_{\text{ind}}) + f(f_{\text{ind}} - d)\Delta s]^2 + z_{R0}^2[(f_{\text{ind}} - d)\Delta s + f_{\text{ind}}d]^2}{z_{R0}f^2f_{\text{ind}}^2}. \end{aligned} \quad (3-4)$$

It can be straightforwardly shown that when  $\Delta s = 0$ , and for  $|f_{\text{ind}}| \gg f$ ,  $w_{\text{obs}} = w_0 d/f$  for large  $d$ . In other words, in the linear case, i.e., when  $f_{\text{ind}} = \infty$ , as well as in the nonlinear case when  $\Delta s = 0$ , the far field width of the Gaussian beam is what is predicted by geometrical optics. Furthermore, from Eq. (3-2), the rate of change of the width with  $\Delta s$  around  $\Delta s = 0$ , (assuming  $z_{R0} \gg f$ , and for large  $d$ ) can be found as

$$\frac{dw_{\text{obs}}}{d\Delta s} = \frac{-2d^2 z_{R0}}{k_0 w_{\text{obs}} f^2 f_{\text{ind}}} = -\frac{w_0 d}{ff_{\text{ind}}} \quad (3-5)$$

provided we invoke the relation for  $w_{\text{obs}}$  derived above. In Eq. (3-5),  $f_{\text{ind}}$  refers to the value of the induced focal length [see Eq. (3-2)] with  $\Delta s = 0$ . Equation (3-5) may be written as

$$\frac{dw_{\text{obs}}}{d\Delta s} = \frac{k_0^4 w_0^5 P L n_2}{2\pi n_0 f^5}. \quad (3-6)$$

This means that in a material with a positive (negative)  $n_2$ , implying a positive (negative) induced lens, the rate of change of the observed width with the position of the nonlinear sample about the focal plane of the external lens will be negative (positive), in agreement with the results from geometrical optics. Furthermore, the rate of change is proportional to the angle of convergence  $w_0/f$  of the externally focused Gaussian, and to the position  $d$  of the observation plane.

In  $z$ -scan, it is often important to relate the *on-axis transmittance* in the far field to the scan distance  $s$  for measurement purposes. Now, for power conservation, the intensity is inversely proportional to the width of the beam in the far field. Hence, a positive  $n_2$  would imply a positive slope as a function of scan distance with increasing  $s$ , and a negative slope for negative  $n_2$ . The value of the slope of the on-axis transmittance could give the information of the  $n_2$  of the material. Note that the  $z$ -scan graph in this case is an *odd* function of the scan distance, measured about the back focal plane of the external lens.

In practice, a detector with a *finite* aperture area is placed on-axis, and the detected power monitored as a function of scan distance. The transmitted power of the Gaussian beam through the aperture is obtained as

$$\begin{aligned} P_T &= \frac{1}{2} \int_0^{2\pi r} \int_0^a \frac{EE^*}{\eta_0} dA = \frac{\pi w_0}{2} \left( \frac{E_0^2}{2\eta_0} \right) \left( 1 - \exp\left(-\frac{2r_a^2}{w_{\text{obs}}^2}\right) \right) \\ &= P \left( 1 - \exp\left(-\frac{2r_a^2}{w_{\text{obs}}^2}\right) \right) \end{aligned} \quad (3-7)$$

where  $r_a$  is the radius of aperture and  $\eta_0$  is the characteristic impedance of free space. The rate of change of  $P_T$  with  $\Delta s$  can be found as

$$\frac{dP_T}{d\Delta s} = \frac{4r_a^2 P}{w_{\text{obs}}^3} \exp\left(-\frac{2r_a^2}{w_{\text{obs}}^2}\right) \frac{dw_{\text{obs}}}{d\Delta s}. \quad (3-8)$$

Substituting Eq. (3-6) into Eq. (3-8) and using the approximation  $w_{\text{obs}} \approx w_0 d/f$ , Eq. (3-8) can be rewritten as

$$\frac{dP_T}{d\Delta s} \cong \frac{4r_a^2 f^2 P}{f_{\text{ind}} d^2 w_0^2} \exp\left(-\frac{2r_a^2 f^2}{d^2 w_0^2}\right), \quad (3-9)$$

from which

$$\frac{dP_T}{d\Delta s} (\Delta s \approx 0) \cong \frac{4k_0^4 w_0^2 \eta_0 r_a^2 P^2 L}{\pi f^2 d^2 n_0} \exp\left(-\frac{2r_a^2 f^2}{d^2 w_0^2}\right) n_2. \quad (3-10)$$

When  $r_a$  approaches 0,

$$\frac{dP_T}{d\Delta s}(\Delta s \approx 0) = 4 \left( \frac{k_0^2 w_0 r_a P}{fd} \right)^2 \frac{\eta_0 L}{\pi n_0} n_2. \quad (3-11)$$

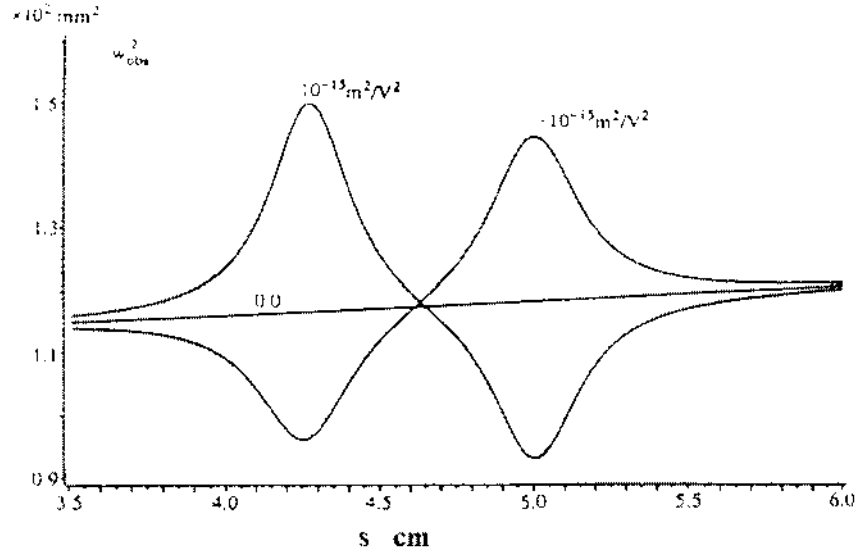
Therefore,  $n_2$  can be estimated as

$$n_2 = \left( \frac{\pi n_0}{n_0 L} \right) \left( \frac{d}{2k_0^2 w_0 P} \right)^2 \left( \frac{f}{r_a} \right)^2 \frac{dP_T}{d\Delta s}. \quad (3-12)$$

Thus, if one knows the slope of the  $z$ -scan graph around  $\Delta s \approx 0$ , the value of  $n_2$  can be found (Banerjee and Pea, 2001).

The situation is not as simple to describe for a thick sample. Indeed, to track the far-field profile in this case, one needs to describe the propagation of the Gaussian beam inside the nonlinear sample using the  $q$  parameter formalism. Because both the width and the radius of curvature of the beam at the exit face of the sample are required to predict the far-field behavior, one can write a simple differential equation describing the variation of the  $q$  through the nonlinear sample. This equation has two parts: the contribution from propagational diffraction through the sample and the effect of the nonlinearity. Explicitly, the differential equation can be written as in Eq. (2-6), where  $a(z)$  and  $w(z)$  denote the on-axis amplitude and width of the Gaussian in the nonlinear material, related through the power-conservation law. If Eq. (2-6) is decomposed into two coupled equations [as in Eqs. (2-10) and (2-12)] describing the evolution of the width and the radius of curvature of the beam through the nonlinear material and solved (Banerjee et al., 1991), we can find the exit widths and radii of curvature for various values of the lens-sample separation  $s$ . For fixed lens-sample separation, if one plots the variation of the width of the Gaussian beam through the nonlinear material, one finds that for a material with a positive  $n_2$ , the width attains a minimum that is smaller than that expected if the sample were linear; furthermore, the location of the waist is to the right of where it would be in the linear case.

Finally, knowing the exit width and radius of curvature of the Gaussian beam from the nonlinear sample, we can compute the width of the Gaussian beam at the observation plane. The results are shown in Fig. 7, taken from Banerjee et al. (1991), where we take the distance  $d$  of the observation plane from the exit face of the nonlinear sample to be 1 m. The nonlinear sample is 1 cm thick and is assumed to have  $n_2 = \pm 10^{-5} \text{ m}^2/\text{V}^2$  (this is a large value for crystals!). For comparison, the case of a linear sample is also superposed on the same graph. Note that with the variation of  $s$ , the far-field width initially increases, and then decreases before starting to increase again, for a material with a negative  $n_2$ . Note that around the point where the variations intersect the graph for the linear case, the slope of the curves are negative and positive for materials with positive and negative  $n_2$ , respec-



**Figure 7** Plot of width of the beam in the far field as a function of the scan distance  $s$ . The light wavelength is 640 nm [Banerjee et al. (1991)].

tively. This is in agreement with the results predicted from geometrical optics and the  $q$  formalism for a thin sample, described earlier.

It is necessary, at this point, to compare the results from the thin and thick sample calculations. Important differences include the position of the points where the graphs for positive and negative  $n_2s$  intersect the graph for the linear case, and the slope of the graphs, predicted from thin and thick sample calculations. For instance, for the thin sample approximation, the value of the slope around  $s = f$  is about  $-4$  for a material with a  $n_2 = 10^{-15} \text{ m}^2/\text{V}^2$ . In this calculation, we have assumed  $L = 1 \text{ cm}$ , and  $n_0 = 1$ . From the graph in the thick sample case, it is, perhaps, difficult to ascertain at exactly which point one must determine the slope. A figure of merit would be the minimum value, which occurs for a value of  $s$  slightly smaller than  $f = 5 \text{ cm}$ . A conservative estimate yields a slope of approximately  $-1$  for this particular example. Note, however, that in the thick sample analysis, we have taken a paraxial approximation to the Gaussian at every stage of its propagation through the nonlinear sample. Strictly, this is not true, and to alleviate this problem, a correction factor is sometimes inserted ad hoc with the width of the Gaussian beam (Sheik-bahae et al., 1989).

The  $z$ -scan method can be also adapted to determine the *nonlinear absorption coefficient*  $\alpha_2$  of a material, defined through the relation  $\alpha = \alpha_0 + \alpha_2 I$ , where  $\alpha$  is the total absorption coefficient and  $I$  denotes the intensity.

Details can be found in Sheik-bahae et al. (1989); suffice to mention here that to achieve this, the aperture is assumed to be completely open in order to collect all the light behind the nonlinear absorptive sample. For  $\alpha_2 > 0$ , maximum absorption is experienced when the nonlinear sample, assumed thin, is at the back focal plane of the lens since the optical intensity is a maximum due to the external focusing. The  $z$ -scan graph is an *even* function of scan distance, measured about the back focal plane of the external lens.

#### 4 SELF-BENDING OF OPTICAL BEAMS

In the previous sections, we have analyzed the effect of cubic nonlinearity on symmetric (Gaussian or sech-type) beams. Symmetric beams induce symmetric self-phase modulation, which focuses or defocuses the beam, but does not change their nominal direction of propagation. On the contrary, an asymmetric beam induces an asymmetric phase modulation profile, which can be responsible for the beam propagating at an angle to its original direction after some distance of propagation. The situation is similar to a wave passing through a prism, resulting in the “bending” of the light path.

To see how this happens in a Kerr medium, let us hypothesize a beam profile  $\psi_{e0}$  in one transverse dimension such that

$$\psi_{e0} = \begin{cases} A^2(1 - x/W), & 0 \leq x \leq W, \\ 0, & \text{otherwise.} \end{cases} \quad (4-1)$$

When this profile travels a “thin” slice of a Kerr material of length  $L$ , the emerging complex field can be expressed as

$$\psi_e(x, L) = \psi_{e0}(x) \exp -jk_0 n_2 A^2 (1 - x/W) L. \quad (4-2)$$

In the far field, one would observe the spatial Fourier transform of the profile, so that

$$\begin{aligned} \psi_e(x, z) &= [\Psi_{e0}(k_x) * \delta(k_x + k_0 n_2 A^2 (1 - x/W) L)] |_{k_x = k_0 x/z} \\ &= \Psi_{e0}((k_0/z)(x + n_2 A^2 L z/w)) \end{aligned} \quad (4-3)$$

where  $\Psi_{e0}(k_x)$  is the spatial Fourier transform of  $\psi_{e0}(x)$ . Hence, looking in the far field, the beam seems to be traveling at an angle of  $-n_2 A^2 L/w$  with respect to its nominal direction of propagation. This is what is termed *self-bending* of the beam, and the amount depends on the nonlinearity and intensity of the beam. The direction of bending depends on the sign of the nonlinearity parameter  $n_2$ . For details, the reader is referred to Swartzlander et al. (1988).

A word of caution: The above simplistic analysis only holds if the thickness of the material is small enough so that there is no appreciable

linear diffraction of the beam in the nonlinear sample. Otherwise, the far-field picture becomes difficult to analyze and can only be performed using a rigorous beam propagation algorithm.

## 5 SELF-PHASE MODULATION DUE TO $\chi^{(2)}$ : EFFECTIVE $\chi^{(3)}$

In the region of intermediate, second-harmonic conversion efficiencies, it has been experimentally observed that the generated second harmonic causes a nonlinearly induced phase change of the fundamental beam (DeSalvo et al., 1991). This has been interpreted as a *cascaded effective third-order nonlinearity* leading to self-focusing or self-defocusing. Although the general analysis of this is rather involved, we will, in this book, derive a very approximate expression for the effective  $\chi^{(3)}$  due to the  $\chi^{(2)}$  in a quadratically nonlinear material and in the presence of linear phase mismatch  $\Delta k$ . A detailed treatment can be found in Kobyakov and Lederer (1996), and references therein.

We start from Eqs. (4-1) and (4-2) in Chap. 3 with  $\rho = 0$  (no walk-off) and  $k_2 \approx 2k_1$  in the second term on the right-hand side, and thereafter expand the  $E_e$ s in terms of a perturbation series (Pliszka and Banerjee, 1993)

$$E_e^{j\omega_0(N)} = \sum_{n=1}^N \tilde{\epsilon}^n a_i^{(n)}(x, y, z); \quad i = 1, 2 \quad (5-1)$$

where the coefficients  $a_i^{(n)}$  are assumed to satisfy amplitude-independent boundary conditions

$$a_i^{(1)}(x, y, 0) = E_e^{\omega_0}(x, y, 0)/\tilde{\epsilon}, \quad (5-2)$$

with all other  $a_i^{(n)}$ s vanishing at  $z = 0$ . The parameter  $\tilde{\epsilon}$  is a bookkeeping parameter. Substituting Eq. (5-1) into Eqs. (4-1) and (4-2) of Chap. 3, we obtain the recursion relations

$$da_1^{(n)}/dz = -j\sigma \exp(-j\Delta kz) \sum_{k=1}^n a_1^{*(k)} a_2^{(n-k)} - j(1/2k_1) \nabla_{\perp}^2 a_1^{(n)} \quad (5-3a)$$

$$da_2^{(n)}/dz = -j\sigma \exp(j\Delta kz) \sum_{k=1}^n a_1^{(k)} a_2^{(n-k)} - j(1/4k_1) \nabla_{\perp}^2 a_2^{(n)}. \quad (5-3b)$$

From Eq. (5-3a), it is easy to see that the fundamental beam to order 2 propagates linearly according to the Fresnel diffraction formula, and

$$E_e^{2\omega_0(1)} = 0. \quad (5-4)$$

For  $E_e^{2\omega_0(2)}$ , we have the linear inhomogeneous equation

$$jdE_e^{2\omega_0(2)}/dz = (1/4k_1) \nabla_{\perp}^2 E_e^{2\omega_0(2)} + \sigma \exp(j\Delta kz) E_e^{2\omega_0(1)^2}. \quad (5-5)$$

It is easy to solve Eq. (5-5) for a Gaussian initial fundamental beam profile, using transform methods. After some straightforward algebra,

$$E_e^{2\omega_0(2)}(x, y, z) = \frac{A(z)}{a(z)} E_e^{2\omega_0(1)^2}, \quad (5-6a)$$

$$a(z) = \frac{1}{1 + (z/z_R)^2} \exp(j \tan^{-1}(z/z_R)), \quad z_R = k_1 w_0^2/2, \quad (5-6b)$$

$$A(z) = -j\sigma \int_0^z \exp(j\Delta k z') a(z') dz'. \quad (5-6c)$$

Note that  $E_e^{2\omega_0(2)} = 0$  at  $z = 0$ , and the width of the second harmonic is  $1/\sqrt{2}$  times that of the fundamental.

Finally, substituting Eq. (5-6a–c) in Eq. (5-3a), we obtain the following equation for the propagation of the fundamental beam, valid up to third-order terms in the expansion parameter  $\tilde{\epsilon}$ :

$$j dE_e^{\omega_0}/dz = (1/2k_1) \nabla_{\perp}^2 E_e^{\omega_0} + \chi^{(3)}(z) |E_e^{\omega_0}|^2 E_e^{\omega_0} \quad (5-7)$$

$$\chi^{(3)}(z) = -j \frac{\exp(-j\Delta k z)}{a(z)} \sigma^2 \int_0^z \exp(j\Delta k z') a(z') dz'. \quad (5-8)$$

Note that  $\chi^{(3)}$  describes an effective nonlinear index of refraction and is proportional to  $\chi^{(2)^2}$ . For weakly focused incident beams,

$$\chi^{(3)}(z) \propto \frac{1 - \exp(-j\Delta k z)}{\Delta k}. \quad (5-9)$$

Observe that the complex nature of  $\chi^{(3)}$  implies that one sees nonlinear absorption as well as nonlinear refraction during the SHG process (DeSalvo et al., 1992). The effective  $\chi^{(3)}$  is proportional to the square of  $\chi^{(2)}$  and inversely proportional to the phase mismatch  $\Delta k$  [De Salvo et al. (1992), Stegeman et al. (1993), Schick (1994)]. Furthermore, the sign of  $\chi^{(3)}$  can be controlled by the sign of the phase mismatch. For this reason, one can expect to see spatial solitons in media with cascaded nonlinearities [Stegeman et al. (1996)]. In fact, bright spatial solitons are also possible in defocusing Kerr media, supported by cascaded nonlinearities [Bang et al. (1997)], since it is possible to get an effective  $\chi^{(3)} > 0$  strong enough to counteract the defocusing effect of the intrinsic  $\chi^{(3)} < 0$ .

## 6 SUMMARY

In this chapter, we have studied self-focusing, self-defocusing, and self-bending of optical beams in a nonlinear material. One- and higher-dimen-

sional spatial soliton solutions have also been found, as solutions of the NLS equation. The phase modulation arising from the intrinsic  $\chi^{(3)}$  of the material, as well as that arising from the combined effects of  $\chi^{(2)}$  and linear phase mismatch in a material during SHG, have been investigated. In the following chapters, we will investigate other effects of  $\chi^{(3)}$ , such as optical bistability, phase conjugation, and soliton propagation in fibers.

## 7 PROBLEMS

1. A sample of cubically nonlinear material has  $n_2 = 10^{-20} \text{ m}^2/\text{V}^2$  and  $n_0 = 1.5$ . Find the values of  $\beta_3$ ,  $\chi^{(3)}$ , and  $\gamma$ , where  $\gamma$  is defined through the relation  $n = n_0 + \gamma I$ , where  $I$  is the intensity.
2. An equation alternately used to model cubic nonlinearity in the presence of dispersion is the Klein–Gordon equation

$$\partial^2 \psi / \partial t^2 - v^2 \nabla^2 \psi = A_1 \psi + A_3 \psi^3.$$

- (a) Putting  $\psi = \text{Re}\{\psi_e(x, y, z, t) \exp j(\omega_0 t - k_0 z)\}$ , derive the PDE for  $\psi_e$ .
- (b) Assuming CW propagation only, find soliton solutions in one transverse dimension.
3. Suppose that the amplitude and the width of the sech solution of the NLS equation are perturbed. Using the second-order differential equation for the width of the beam, derive the differential equation (linearized) that describes the behavior of the perturbations. Hence show that the solutions are oscillatory in the direction of propagation, and find the period of the perturbations.
4. A Gaussian beam with plane wave fronts is incident on a 1-cm-thick sample of a nonlinear material of  $n_0 = 1$ ,  $n_2 = 10^{-16} \text{ m}^2/\text{V}^2$ . The beam power is 1 W and the intensity half-width is 0.5 mm. Find the exit width and radius of curvature of the beam.
5. Derive the differential equations that describe the evolution of the widths  $w_x$ ,  $w_y$  of an elliptic Gaussian beam during propagation through an  $n_2$  medium.
6. From the NLS equation, show that during the propagation of a beam, there is conservation of power. To show this, prove that

$$\int_{-\infty}^{\infty} |\psi_e|^2 dx = \text{const.}$$

7. A Gaussian beam of intensity waist 1 mm falls on a knife-edge that eliminates half of the profile. The beam is now incident on a 1-mm-thick sample of a cubically nonlinear material with  $n_0 = 1$ ,

- $n_2 = 10^{-16} \text{ m}^2/\text{V}^2$ . The initial power of the beam is 1 W. Find the approximate angular deflection of the beam in the far field.
8. Plot the real and imaginary parts of the effective  $\chi^{(3)}$  in a  $\chi^{(2)}$  material with linear phase mismatch  $\Delta k$ . Show that the absorption drops off much faster with  $\Delta k$  than the nonlinear phase shift.
  9. Two Gaussian beams travel side by side in a  $\chi^{(3)}$  medium. Assuming that they are mutually coherent and in phase, discuss under what conditions they will attract each other and form one beam.

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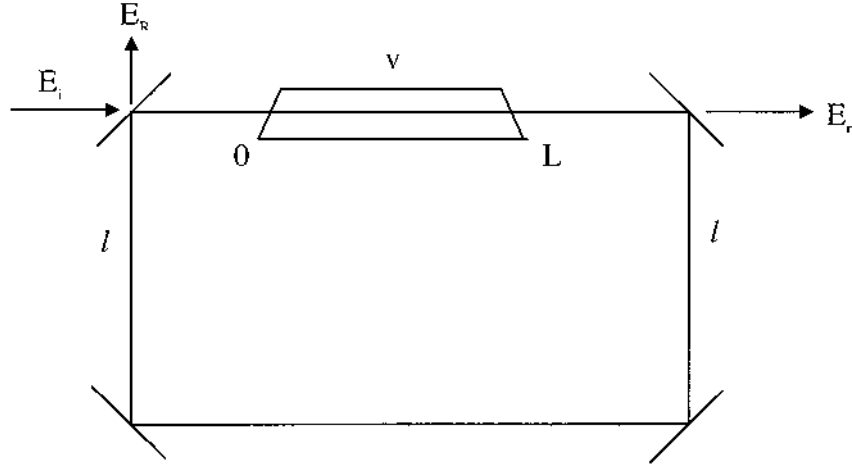
# 5

## Optical Bistability

In the last chapter, we examined the effect of cubic nonlinearity on self-refraction of an optical beam. We have also seen that the presence of a quadratic nonlinearity can also contribute to change in the profile of a Gaussian beam in a crystal. In this chapter we will see how the effect of nonlinearity, along with the proper feedback, can give rise to *optical bistability* and *hysteresis*. This is not unexpected as hysteresis and bistability are also observed in nonlinear electronic circuits with feedback, such as in the *Schmitt trigger*, as well as in *hybrid optical devices*, such as an acoustooptic device with feedback (Banerjee and Poon, 1991). In what follows, we will examine two different types of optically bistable devices based on optical nonlinearities. The first is a *nonlinear ring resonator* comprising a two-level gain medium as the nonlinear element. We will treat both *absorptive* and *dispersive* bistability. *Absorptive bistability* is the case when the incident optical frequency is close to or equal to the transition frequency of the atoms from one level to another. In this case, the absorption coefficient becomes a nonlinear function of the incident intensity. On the other hand, if the frequencies are far apart, the medium behaves like a Kerr-type material and the system exhibits what is called *dispersive bistability*. In this case the material can be modeled by an effective  $n_2$  or  $\chi^{(3)}$ , meaning that the refractive index becomes a nonlinear function of the optical intensity. The second configuration we will examine is a *linear–nonlinear interface*. In this case, the interface provides the feedback due to refractive index mismatch. In fact, the reflection coefficient (as well as the transmission coefficient) is modified due to the nonlinearity, which, as we shall see later, is responsible for demonstrating optical switching and bistability.

### 1 THE NONLINEAR RING CAVITY

Fig. 1 shows a schematic of a ring cavity in which the top arm is assumed to have the nonlinear material, in this case a two-level system. Before proceeding



**Figure 1** Schematic of a nonlinear ring cavity. The active two-level system is in the upper arm. The top mirrors, 1 and 2, have finite reflectivities and transmittivities  $R$  and  $T$  respectively, while the bottom two mirrors, 3 and 4, have 100% reflectivity.

any further, we need to characterize the two-level atomic system and establish the nonlinear response of the system for the cases mentioned in the Introduction.

Optical propagation through a nonlinear two level atomic system is modeled through the *Maxwell–Bloch equations* (Lugiato, 1984). This is a set of three coupled nonlinear equations involving the electric (optical) field  $E_e$ , the polarization  $P_e$ , and a parameter  $\tilde{D}$  which is defined as one half of the difference in population between the upper and lower levels. These equations are

$$\frac{\partial E_e}{\partial t} + c_0 \frac{\partial E_e}{\partial z} = -gP_e, \quad (1-1)$$

$$\frac{\partial P_e}{\partial t} = \left( \frac{\tilde{\mu}}{\hbar} \right) E_e \tilde{D} - j\Delta P_e, \quad (1-2)$$

$$\frac{\partial \tilde{D}}{\partial t} = -\left( \frac{\tilde{\mu}}{2\hbar} \right) (E_e P_e^* + E_e^* P_e) - \gamma_1 \left( \tilde{D} - \frac{N}{2} \right). \quad (1-3)$$

In Eqs. (1-1)–(1-3),

- $N$ : number of participating two level atoms
- $\tilde{\mu}$ : modules of the dipole moment of the atoms
- $g$ : coupling constant
- $\Delta = (\omega_a - \omega_0) - j\gamma_2$
- $\omega_a$ : transition frequency of the atoms

$\omega_0$ : optical frequency  
 $\gamma_{1,2}$ :  $1/T_{1,2}$  where  $T_{1,2}$  are characteristic relation times  
 $c_0$ : light velocity in vacuum  
 $\hbar = h/2\pi$ , where  $h$  is Planck's constant.

The first equation describes unidirectional propagation of a plane wave in the presence of a source term which is proportional to the polarization  $P_e$ . The third equation shows that the rate of change of the population difference depends on a generation term which indicates the action of the optical field on the material, and a decay term due to relaxation of atoms from the upper to the lower level. The second equation shows the temporal dynamics of the material polarization, this depends on the strength of the resonant interaction between the field and the two-level ensemble, which in turn depends on the number of participating atoms  $\tilde{D}$  and is also proportional to the Rabi frequency ( $\omega_a$ ). The formal derivation of the *Bloch equations* can be performed by considering the time evolution of the *density operator* for the ensemble of entities or states (a collection of atoms). The density operator  $\tilde{\rho}$  is defined as  $\tilde{\rho} = \sum_n p_n P(\psi_n)$  where  $p_n$  is the probability that the system is in state  $\psi_n$  and  $P(\psi)$  is the *projection operator* for the state  $\psi$ , defined by  $P(\psi)\psi' = \psi \int d\zeta \psi^* \psi'$ , where  $\psi'$  is an arbitrary wave function. The evolution of  $\tilde{\rho}$  is given by the relation  $j\hbar d\tilde{\rho}/dt = [H, \tilde{\rho}]$ , where  $H$  is the *Hamiltonian* of the system. Details of this derivation are outside the scope of this book, and we refer readers to Butcher and Cotter (1990).

In the steady state, Eqs. (1-1)–(1-3) become

$$\frac{dE_e}{dz} = -\frac{g}{c_0} P_e, \quad (1-4)$$

$$\frac{\tilde{\mu}}{\hbar} E_e \tilde{D} = j[(\omega_a - \omega_0) - j\gamma_2] P_e, \quad (1-5)$$

$$\frac{\mu}{2\hbar} (E_e P_e^* + E_e^* P_e) + \gamma_1 \left( \tilde{D} - \frac{N}{2} \right) = 0. \quad (1-6)$$

Substituting Eq. (1-5) in Eq. (1-6) and solving for  $\tilde{D}$  in terms of  $E_e$ , we get

$$\tilde{D} = \frac{\gamma_1 N/2}{\gamma_1 + \frac{(\tilde{\mu}/\hbar)^2 \gamma_2 |E_e|^2}{(\omega_a - \omega_0)^2 + \gamma_2^2}}. \quad (1-7)$$

Hence from Eq. (1-5),

$$P_e = \frac{-j \left( \frac{\tilde{\mu}}{\hbar} \right) \gamma_1 (N/2) E_e}{\{(\omega_a - \omega_0 - j\gamma_2)\} \left\{ \gamma_1 + \frac{(\mu/\hbar)^2 \gamma_2 |E_e|^2}{(\omega_a - \omega_0)^2 + \gamma_2^2} \right\}}. \quad (1-8)$$

Substituting in Eq. (1-4), we can write

$$\frac{dE_e}{dz} = (-\alpha - jk_0 n)E_e, \quad (1-9)$$

where

$$\alpha = \alpha(|E_e|^2) = \frac{\frac{g}{c_0} \left(\frac{\tilde{\mu}}{\hbar}\right) \gamma_1 \frac{N}{2} \gamma_2}{\gamma_1 [(\omega_a - \omega_0)^2 + \gamma_2^2] + \left(\frac{\tilde{\mu}}{\hbar}\right)^2 \gamma_2 |E_e|^2} \quad (1-10)$$

$$n = n(|E_e|^2) = \frac{\frac{g}{c_0} \left(\frac{\tilde{\mu}}{\hbar}\right) \gamma_1 \frac{N}{2} (\omega_a - \omega_0)/k_0}{\gamma_1 [(\omega_a - \omega_0)^2 + \gamma_2^2] + \left(\frac{\tilde{\mu}}{\hbar}\right)^2 \gamma_2 |E_e|^2}. \quad (1-11)$$

Note that  $\alpha$  and  $n$  include effects of nonlinear absorption and refractive index change, respectively.

### 1.1 Absorptive Bistability

In this case,  $\omega_0 = \omega_a$ , so that

$$\alpha = \frac{\frac{g}{2c_0} \left(\frac{\tilde{\mu}}{\hbar}\right) N \gamma_1}{\gamma_1 \gamma_2 + \left(\frac{\tilde{\mu}}{\hbar}\right)^2 E_e^2}. \quad (1-12)$$

Now, putting

$$\tilde{\psi}_e = \tilde{\mu} E_e / \hbar (\gamma_1 \gamma_2)^{1/2} \quad (1-13)$$

in Eq. (1-9) with Eq. (1-12), we can derive, after some algebra,

$$\frac{d\tilde{\psi}_e}{dz} = -\frac{\alpha_c \tilde{\psi}_e}{1 + \tilde{\psi}_e^2} \quad (1-14)$$

where we have taken  $\tilde{\psi}_e$  to be real, without loss of generality. Upon integrating, we have

$$\ell n \tilde{\psi}_e + \frac{1}{2} \tilde{\psi}_e^2 = -\alpha_c z + \text{const.} \quad (1-15)$$

The constant of integration can be determined by putting  $\tilde{\psi}_e = \tilde{\psi}_e(0)$  at  $z=0$ . Using this, the value of  $\tilde{\psi}_e$  at  $z=L$  can be found from the equation

$$\ell n(\tilde{\psi}_e(0)/\tilde{\psi}_e(L)) + \frac{1}{2} [\tilde{\psi}_e^2(0) - \tilde{\psi}_e^2(L)] = \alpha_c L, \quad (1-16)$$

where  $L$  is the length of the active medium.

It is now time to examine the boundary conditions. These can be written as

$$E_{\text{et}} = T^{1/2} E_{\text{e}}(L) \quad (1-17a)$$

$$E_{\text{et}}(0) = T^{1/2} E_{\text{i}} + R E_{\text{e}}(L); \quad \text{t : transmitted, i : incident.} \quad (1-17b)$$

For simplicity, we have assumed the round trip phase shift in the absence of nonlinearity to be an integral multiple of  $2\pi$ . We now normalize Eq. (1-17a) using

$$\tilde{\psi}_{\text{ei,t}} = \tilde{\mu} E_{\text{ei,t}} / \hbar (\gamma_1 \gamma_2 T)^{1/2} \quad (1-18)$$

which translates the above boundary conditions to

$$\begin{aligned} \tilde{\psi}_{\text{et}} &= \tilde{\psi}_{\text{e}}(L), \\ \tilde{\psi}_{\text{e}}(0) &= T \tilde{\psi}_{\text{ei}} + R \tilde{\psi}_{\text{et}}. \end{aligned} \quad (1-19)$$

From Eqs. (1-16) and (1-19), we can express  $\tilde{\psi}_{\text{et}}$  in terms of  $\tilde{\psi}_{\text{ei}}$  as

$$\begin{aligned} \ell n[1 + T(\tilde{\psi}_{\text{ei}}/\tilde{\psi}_{\text{et}} - 1)] + (\tilde{\psi}_{\text{et}}^2/2)\{[1 + T(\tilde{\psi}_{\text{ei}}/\tilde{\psi}_{\text{et}} - 1)]^2 - 1\} \\ = \alpha_{\text{c}} L \end{aligned} \quad (1-20)$$

Assume  $\alpha_{\text{c}} L \ll 1$ ,  $T \ll 1$  with the *mean-field limit*

$$\frac{\alpha_{\text{c}} L}{2T} = C = \text{constant}; \quad (1-21)$$

then Eq. (1-20) can be approximated to give

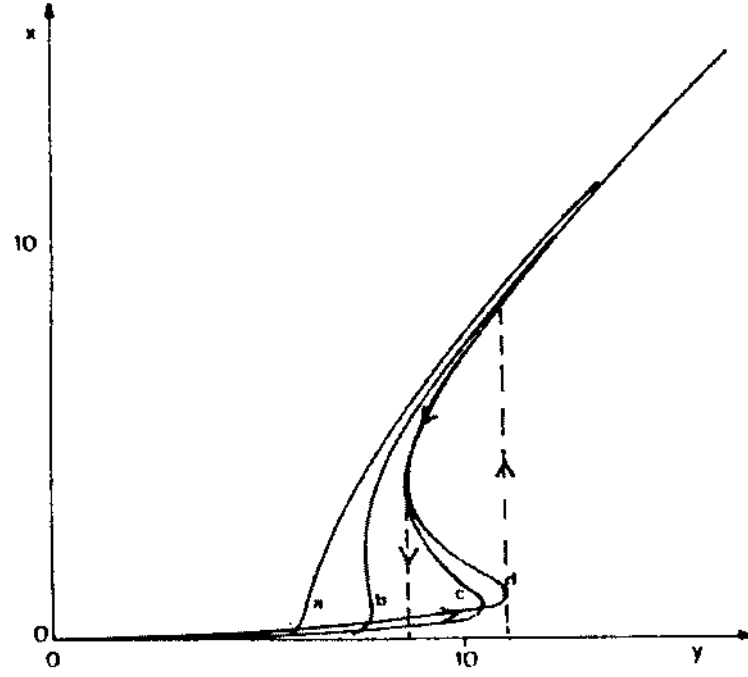
$$\tilde{\psi}_{\text{ei}} = \tilde{\psi}_{\text{et}} + 2C \frac{\tilde{\psi}_{\text{et}}}{1 + \tilde{\psi}_{\text{et}}^2}. \quad (1-22)$$

The plot of  $\tilde{\psi}_{\text{et}}(x)$  vs.  $\tilde{\psi}_{\text{ei}}(y)$  is shown in Fig. 2 for various values of  $\alpha_{\text{c}} L$ , and also in the mean-field limit. Note the *multivalued* nature of some of the graphs, this implies *bistable behavior*. For instance, as shown in curve d in Fig. 2, as  $\tilde{\psi}_{\text{ei}}$  increases, the output  $\tilde{\psi}_{\text{et}}$  remains in the lower branch and then switches to the upper branch when  $\tilde{\psi}_{\text{ei}}$  increases above a certain threshold  $\tilde{\psi}_{\text{ei}2} > 10$  and remains in the upper branch. Upon reducing  $\tilde{\psi}_{\text{ei}}$ , the  $\tilde{\psi}_{\text{et}}$  remains in the upper branch and switches to the lower branch when  $\tilde{\psi}_{\text{ei}}$  decreases below a certain threshold  $\tilde{\psi}_{\text{ei}1} < 10$ . This effect is termed *hysteresis* and is commonly observed in optically bistable systems.

## 1.2 Dispersive Bistability

In this case,  $\omega_0$  is assumed to be far from  $\omega_{\alpha}$ , and  $n \neq 0$ . Focusing only on the contribution from  $n$  on  $E_{\text{e}}$ , we can first write, from Eq. (1-11),

$$n = \tilde{n}_0 + n_2 |E_{\text{e}}|^2, \quad (1-23)$$



**Figure 2** Steady state characteristics for different increasing values of  $\alpha_c L$  (a,b,c), and in the mean-field limit. The arrows show the hysteresis plot of the output as the input is increased and then decreased [Lugiato (1984)].

where

$$\tilde{n}_0 = \frac{-\frac{g}{c_0} \left( \frac{\tilde{\mu}}{\hbar} \right) \left( \frac{N}{2} \right) \left( \frac{\omega_a - \omega_0}{k_0} \right)}{\left[ (\omega_a - \omega_0)^2 + \gamma_2^2 \right]}, \quad (1-24a)$$

$$n_2 = \frac{\frac{g_0}{c_0} \left( \frac{\tilde{\mu}}{\hbar} \right)^3 \gamma_2 \left( \frac{N}{2} \right) \left( \frac{\omega_a - \omega_0}{k_0} \right)}{\gamma_1 \left[ (\omega_a - \omega_0)^2 + \gamma_2^2 \right]^2}. \quad (1-24b)$$

Returning to Eq. (1-9) and assuming only the contribution from  $n$ , we can now set

$$E_e(z) = |E_{e0}| e^{-j\phi(z)} \quad (1-25)$$

so that

$$\frac{d\phi}{dz} = k_0 n(|E_e|^2). \quad (1-26)$$

Now, we write

$$\begin{aligned} I_i &= E_{ei}^2 \text{ (assumed real).} \\ I_t &= |E_{et}|^2, \end{aligned} \quad (1-27)$$

along with boundary conditions

$$E_{er} = \sqrt{T}E_e(L), \quad (1-28a)$$

$$E_{e0} = \sqrt{T}E_{ei} + Re^{-j\delta_0}E_e(L). \quad (1-28b)$$

In Eq. (1-28b),  $\delta_0$  is the linear cavity detuning defined as

$$\delta_0 = \frac{\omega_c - \omega_0}{c/\tilde{L}}, \quad \tilde{L} = 2(L + \ell) \quad (1-29)$$

where  $\omega_c$  is the frequency of the cavity nearest to resonance with the incident field. An equivalent set of boundary conditions can be written down in terms of the phasor electric field as well.  $\tilde{L}$  denotes the total optical path length of the cavity; the linear refractive index of the cavity can be assumed to be unity for simplicity. The transmission of the cavity  $\mathfrak{T}$  can be written as

$$\mathfrak{T} = \frac{I_t}{I_i} = \frac{T^2}{T^2 + 4R \sin^2 \left\{ \frac{1}{2} (\phi(L) - \phi(0) - \delta_0) \right\}} \quad (1-30)$$

and resembles the traditional transmission of a ring cavity. Note that for an empty cavity,

$$\mathfrak{T} = \frac{1}{1 + \frac{4R}{T^2} \sin^2 \frac{\delta_0}{2}}, \quad (1-31)$$

which defines the resonance curve of a linear ring cavity or a Fabry–Perot cavity.

In the *dispersive* case,

$$\mathfrak{T} = \frac{I_t}{I_i} = \frac{1}{1 + \frac{4R}{T^2} \sin^2 \left\{ \frac{1}{2} (\delta + \beta I_t) \right\}} \quad (1-32)$$

where

$$\begin{aligned} \delta &= \delta_0 + k_0 L \tilde{n}_0, \\ \beta &= k_0 L n_2 / T. \end{aligned} \quad (1-33)$$

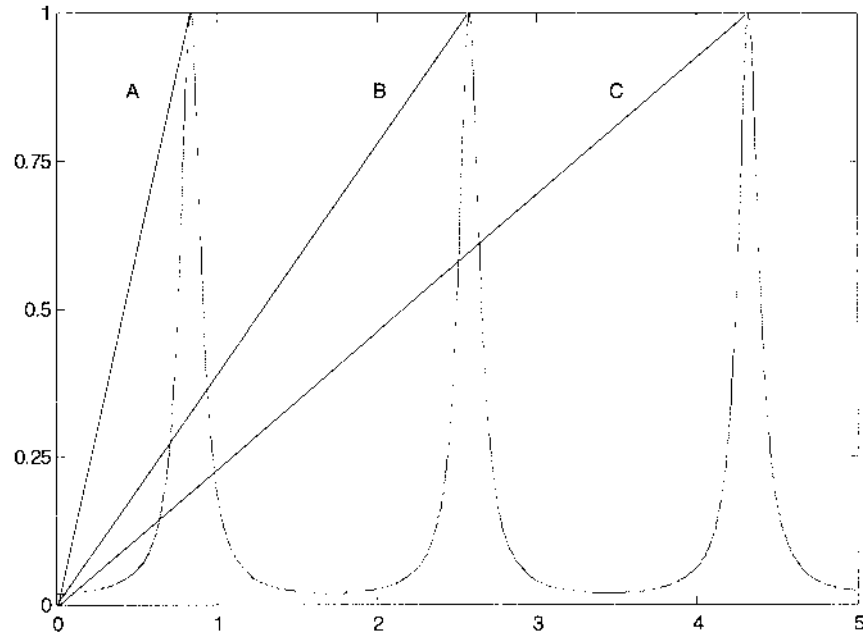
Eqs. (1-32) and (1-33) define the steady-state input–output relationship for a nonlinear Kerr-type ring cavity.

Fig. 3 shows how one can graphically solve Eq. (1-32). The plot of  $\mathfrak{S}$  vs.  $I_t$  using the relation  $\mathfrak{S} = I_t/I_i$  is a straight line. Also superposed is the plot of

$$\mathfrak{S} = \frac{1}{1 + \frac{4R}{T^2} \sin^2 \left\{ \frac{1}{2} (\delta + \beta I_t) \right\}}$$

vs.  $I_t$ , and this is similar to the resonance plot of a cavity. The point(s) of intersection of these two plots gives the solution of Eq. (1-32). Note that for line B, there are three intersecting points away from zero and hence three steady-state solutions. Let us call three points  $a$ ,  $b$ , and  $c$  for convenience, in order of increasing  $I_t$ .

The stability of the steady-state solutions  $a$ ,  $b$ ,  $c$  can be heuristically argued in the following way. Consider the point  $c$  at the second peak. If  $I_t$  slightly increases (due to a small perturbation in the system), the cavity transmission should decrease, leading to a decrease in  $I_t$ . Hence point  $c$  is a *stable* operating point.

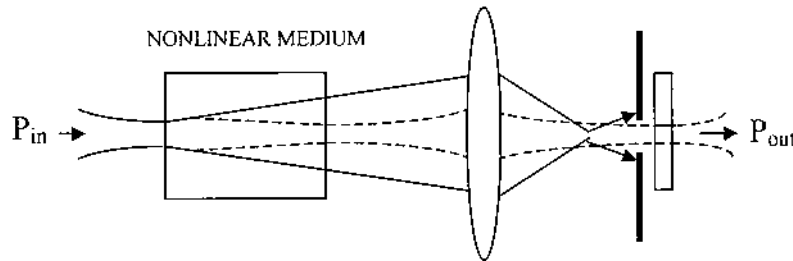


**Figure 3** Graphical derivation of the steady state for dispersive bistability. The horizontal axis is  $I_t$  and the vertical axis is  $\mathfrak{S}$ .

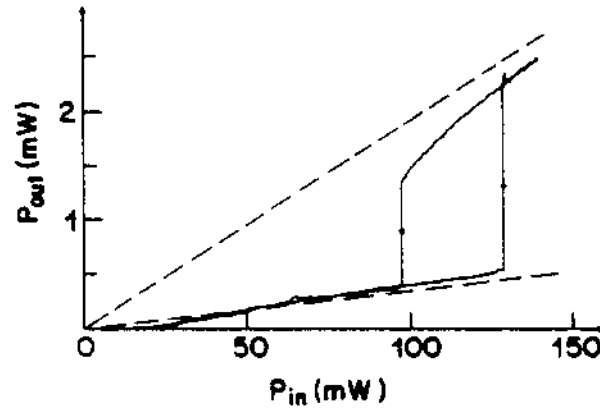
## 2 TRANSVERSE OPTICAL BISTABILITY

The previous section described a ring cavity with the active arm filled with a two-level system which could exhibit absorptive or dispersive bistability depending on the input frequency and the transition frequency of the two-level system. In this section, we give an example of bistability in a system without an optical cavity. The absence of the cavity eliminates cavity build-up and decay time constants and relaxes the bandwidth requirements of the laser. The work on this kind of bistable system, which relies instead on the self-focusing or self-defocusing of optical beams, is called *transverse bistability* (see Gibbs, 1985) and was first introduced by Bjorkholm et al. (1982).

The idea of *transverse bistability* based on the formation of a *self-trapped filament* is shown in Fig. 4. If the input Gaussian beam is focused to an appropriate diameter on the input face of an appropriate self-focusing medium, there is a critical value  $P_{cr}$  of the input power  $P_{in}$  for which the input laser beam passes through the medium with little change in diameter, i.e., it is self-trapped (see Chap. 4 for details). For bistability, a lens L, aperture A, and mirror M are added such that there is good feedback for the self-trapped output but poor feedback for the low-intensity ( $P_{in} < P_{cr}$ ) propagation. As explained in Gibbs (1985), *hysteresis* comes from the fact that once trapping is established,  $P_{in}$  can be reduced below  $P_{cr}$  and trapping will be maintained so long as  $(1 + 2R_{eff}) P_{in} \geq P_{cr}$ , where  $R_{eff}$  is the effective feedback reflectivity into the self-trapped filament. [The factor of 2 arises because the counterpropagating field shifts the phase twice as much as the self-shift of the forward field (see Gibbs, 1985). Rapid diffusion can remove the grating between the forward and backward fields and reduce the factor of 2 to 1.] The essential phenomenon in *self-trapping bistability* is the movement of the apparent focus from the entrance to the exit of a long self-focusing medium by self-trapping.



**Figure 4** Schematic diagram for setup to demonstrate transverse optical bistability. The lens, L, is followed by the aperture, A, and a mirror, M.



**Figure 5** Experimental curve of output vs. input power for self-trapping bistable operation [Bjorkholm et al. (1981)].

Using Na vapor as the nonlinear material, an initial laser beam width of approximately 100  $\mu\text{m}$ , Bjorkholm et al. (1981) have observed optical bistability as shown in Fig. 5. Without the feedback mirror, self-focusing and self-trapping were observed with about 150 mW input optical power at a frequency that would generate a nonlinear Kerr-type effect in Na.

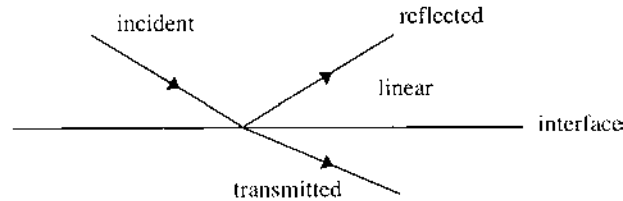
Kaplan (1981) has proposed the elimination of *all* feedback mirrors by using counterpropagating beams whose mutual influence by way of the nonlinear refractive index provides the feedback essential for bistability. His schemes can even work for defocusing, thus avoiding a catastrophic collapse of the beam in the case of an unsaturable nonlinear material. Defocusing schemes allow the use of semiconductors whose nonlinear refractive indices are large and negative just below the band edge where the absorption is sufficiently low. Kaplan also suggests bistability by using the self-bending of a light beam whose intensity profile possesses an asymmetrical shape.

For a more detailed description and summary of transverse optical bistability, the reader is referred to Gibbs (1985).

### 3 LINEAR–NONLINEAR INTERFACE

#### 3.1 Grazing Incidence

Another example of cavity-less operation is the linear–nonlinear interface, first proposed by Kaplan (1977). In this setup, shown in Fig. 6, light is incident at a large angle of incidence onto the interface between a linear and a nonlinear medium. The refractive index of the nonlinear medium is intensity-dependent

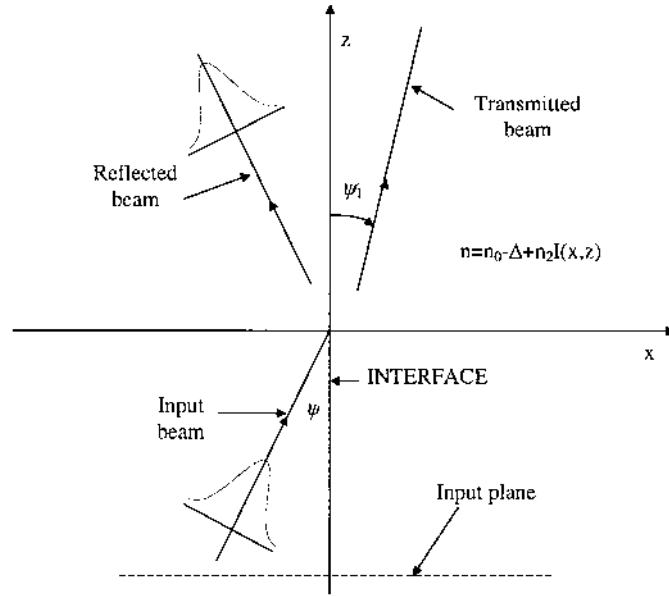


**Figure 6** Schematic setup of linear/nonlinear interface with input plane wave at grazing incidence.

due to the Kerr effect and it is assumed that the refractive indices of the linear and nonlinear medium are close to each other. The principal effect predicted is that when the glancing incident angle or the incident field intensity is varied, strong hysteresis jumps should be observed in the reflection coefficient from the interface. The modified Snell's laws that follow from the boundary conditions for the angle of refraction and the Fresnel laws of reflection and transmission across the interface are now coupled to one another because of the nonlinearity of the Kerr medium. In the case of total internal reflection, a surface wave propagates inside the nonlinear refractive medium along the boundary. The analysis shows the appearance of several physically realizable solutions (or states), some of which are unstable and this, in fact, is the reason for *hysteresis*. As seen earlier, systems exhibiting hysteresis have strong marked thresholds. Optical elements with thresholds are potentially useful for optical switching and logic, and because the nonlinear interface does not involve any resonator, it offers the possibility of ultrafast switching.

Kaplan's theory of bistability and hysteresis across a linear–nonlinear interface at glancing incidence seems to agree with experimental results (Smith et al., 1981). Unfortunately, these experimental results have been hard to interpret. The existing line of thought is that in any experiment, the input beam will have a finite width. The problem cannot be analytically tackled and therefore has to be studied numerically. The geometry of the problem is shown in Fig. 7. The angle  $\psi$  was taken as  $5^\circ$  for computation, and the low-intensity critical angle is about  $9^\circ$ . For details of the simulation, the reader is referred to Smith et al. (1981).

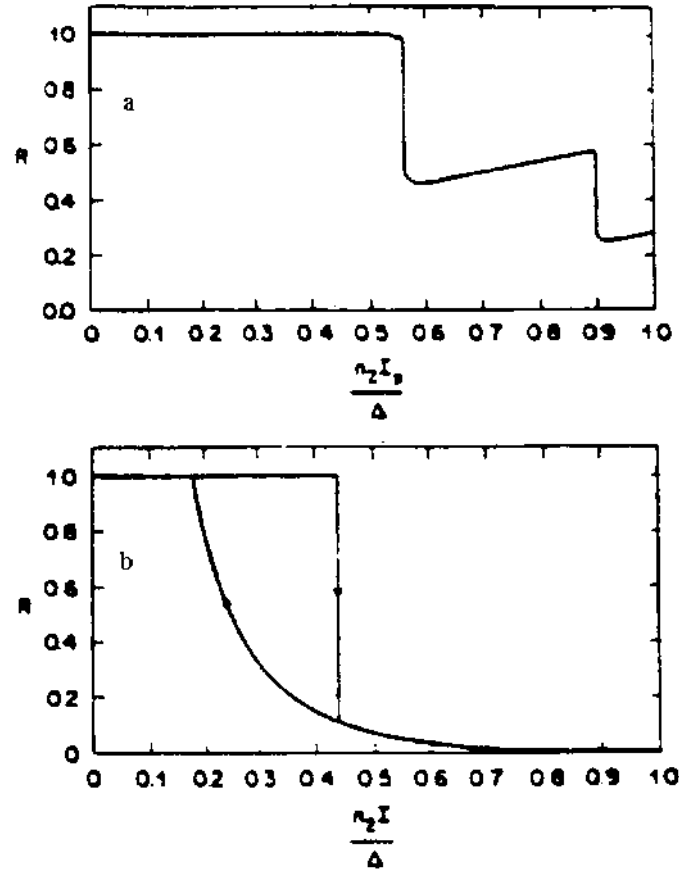
As the intensity of the input beam is increased from zero, the first consequence of the nonlinearity is a large nonlinear *Goos–Hanchen shift*. For those readers not familiar with the Goos–Hanchen effect, this refers to a displacement of the reflected beam parallel to the interface with respect to the incident beam (Banerjee and Poon, 1991). This is a consequence of the phase shift during total internal reflection and is a function of the angle of incidence. At higher intensities, light penetrates into the nonlinear medium and prop-



**Figure 7** Linear/nonlinear interface with Gaussian beam incident from the linear medium.

agates in a narrow self-focussed channel. Results of the reflectivity from the interface are plotted as a function of the intensity of the beam, and the results are compared with the predictions of the plane wave theory from Kaplan (1977). The curve (see Fig. 8a) displays the sharp drop in reflectivity at a threshold intensity; this occurs when light penetrates into the nonlinear medium. For comparison, Fig. 8b shows the same plot as predicted by plane wave theory for a plane wave incident at the same angle as the Gaussian beam. The major difference between the two graphs is the lack of hysteresis in Fig. 8a. However, both predict sharp thresholds which may be important in finding applications in optical switching.

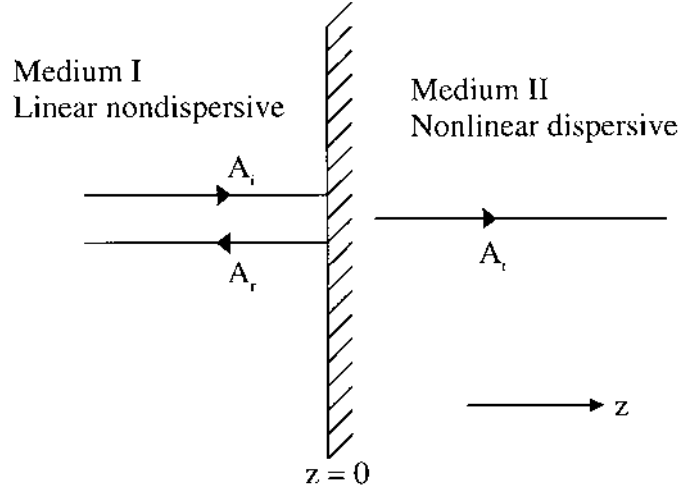
In closing, it should be pointed out that Smith et al. (1979) observed hysteresis experimentally using pulsed laser at a glass-CS<sub>2</sub> boundary. It is now believed that in effect, no hysteresis occurs, and the observed effect is a consequence of the fact that the lower branch of the reflection hysteresis curve is metastable, and can persist for hundreds of relaxation times, thus explaining the glass-CS<sub>2</sub> hysteresis. The prediction of hysteresis from Kaplan's theory apparently arises from the fact that the incoming plane wave may excite surface waves, so that the plane-wave approximation breaks down [Tomlinson et al. (1982)].



**Figure 8** Comparison of (a) Gaussian beam analysis to (b) plane wave analysis for a linear-nonlinear interface. Note the absence of hysteresis in (a).

### 3.2 Normal Incidence: Linear Nondispersive–Nonlinear Dispersive Interface

Consider the situation as shown in Fig. 9, where a plane wave is normally incident at the boundary of a linear nondispersive–nonlinear dispersive interface. In this case, the laws of reflection and refraction are trivial; however, the Fresnel relations for the reflection and transmission coefficient are modified. The analysis presented here closely follows the treatment in Cao and Banerjee (1986), and predicts hysteresis and bistability similar to the effects predicted by Kaplan (1977).



**Figure 9** Schematic for normal incidence at a linear nondispersive/nonlinear dispersive interface.

### 3.2.1 Solutions for the Propagation Case

In electromagnetics we know that for normal incidence at an interface of two semifinite regions with characteristic impedances  $\eta_1$  and  $\eta_2$ , the transmission coefficient  $t$  at the interface for a time-harmonic wave with polarization parallel to the interface is given by

$$t = \frac{2\eta_2}{\eta_1 + \eta_2}, \quad (3-1)$$

with the reflection coefficient  $r$  given by

$$r = t - 1. \quad (3-2)$$

The characteristic impedance  $\eta$  may easily be found by using the simple definitions (see Chap. 1).

$$\eta = \left(\frac{\mu}{\varepsilon}\right)^{1/2} = \frac{\omega_0 \mu}{k_0}, \quad (3-3)$$

where  $\mu$  and  $\varepsilon$  denote the permeability and permittivity of the medium, respectively, and  $\omega_c$  and  $k_0$  denote, respectively, the angular frequency and propagation constant of the wave. In our case we can analogously define an effective characteristic impedance  $\eta_t$  in terms of an effective propagation constant  $k_t$  as

$$\eta_t = \frac{\omega_0 \mu}{k_t} \quad (3-4)$$

for time-harmonic waves.

We will now determine the *effective propagation constant*  $k_t$  (and hence the effective characteristic impedance  $\eta_t$ ) in a nonlinear dispersive medium.

We choose to use the *nonlinear Klein–Gordon equation* (Korpel and Banerjee, 1984)

$$\frac{\partial^2 \psi}{\partial t^2} - \tilde{c}_0^2 \frac{\partial^2 \psi}{\partial z^2} = -\omega_c^2 \psi + \frac{2}{3} \beta_3 \frac{\partial^2 \psi^3}{\partial t^2} \quad (3-5)$$

over the more conventional *nonlinear Schrodinger (NLS) equation* in order to model the wave propagation in medium II with a (linear) dispersion relation which can be explicitly expressed in the form

$$\omega^2 = \tilde{c}_0^2 k^2 + \omega_c^2. \quad (3-6)$$

where  $\omega_c$  denotes the cutoff frequency.

We now substitute

$$\psi(z, t) = \text{Re} [\psi_p(z) \exp(j\omega_0 t)] \quad (3-7)$$

where  $\omega_0$  is the carrier frequency, into Eq. (3-5) to obtain

$$\frac{d^2 \psi_p}{dz^2} + \tilde{k}_0^2 \psi_p - k_c^2 \psi_p - \frac{1}{2} \beta_3 \tilde{k}_0^2 |\psi_p|^2 \psi_p = 0, \quad (3-8)$$

where  $\tilde{k}_0 = \omega_0/\tilde{c}_0$ , and when a weak nonlinearity is assumed. Eq. (3-8) describes the propagation of the phasor optical field along a medium with dispersion and a cubic nonlinearity.

A particular solution of Eq. (3-8) is of the form

$$\psi_p(z) = A_t \exp(-jk_t z) \quad (A_t \text{ constant}), \quad (3-9)$$

and

$$k_t = \pm \frac{1}{\tilde{c}_0} \left( \omega_0^2 - \omega_c^2 - \frac{1}{2} \beta_3 \omega_c^2 |A_t|^2 \right)^{1/2}. \quad (3-10)$$

In Eq. (3-10),  $k_t$  represents the propagation constant in the nonlinear dispersive medium (II). Now, by taking  $k_0$  as the propagation constant of the incident wave in the linear medium (I), the transmission coefficient  $T$  for the linear nondispersive ( $z < 0$ )–nonlinear dispersive ( $z > 0$ ) interface may be derived, by using Eqs. (3-1), (3-3), (3-4), and (3-10) to be

$$T = 2 \left/ \left\{ 1 \pm \frac{\tilde{k}_0}{k_0} \left[ 1 - \left( \frac{\omega_c}{\omega_0} \right)^2 - \frac{1}{2} \beta_3 |A_t|^2 \right]^{1/2} \right\} \right. \quad (3-11)$$

If

$$1 - \left( \frac{\omega_c}{\omega_0} \right)^2 - \frac{1}{2} \beta_3 |A_i|^2 |T|^2 < 0, \quad (3-12)$$

then the propagation constant  $k_t$  is imaginary and the transmission coefficient  $T$  is complex.  $k_t$  is given by Eq. (3-10) as

$$k_t = -j\tilde{k}_c \left[ 1 - \left( \frac{\omega_c}{\omega_0} \right)^2 - \frac{1}{2} \beta_3 |A_i|^2 |T|^2 \right]^{1/2} \quad (3-13)$$

where the minus is chosen to ensure a decaying mode. However, Eq. (3-13) is valid only at  $z=0^+$ . Hence we will have to find an approach to obtain a solution for  $z > 0$  in this nonpropagating case.

For expression (3-12) (the condition for nonpropagation) to be satisfied, we may have (1) medium II linear dispersive and the operating frequency below cutoff, (2) medium II nonlinear nondispersive with a positive nonlinearity strong enough that  $\beta_3 |A_i|^2 / 2 > 1/|T|^2$ , (3) medium II nonlinear dispersive and (a) the operating frequency below cutoff with a positive nonlinearity or a weak nonlinearity  $\beta_3 < 0$  such that  $|\beta_3| |A_i|^2 / 2 < [(\omega_c/\omega_0)^2 - 1]/|T|^2$ , (b) the operating frequency above cutoff with  $\beta_3 > 0$  (or  $n_2 < 0$ ) so that

$$\frac{1}{2} |\beta_3| |A_i|^2 > \left[ 1 - \left( \frac{\omega_c}{\omega_0} \right)^2 \right] / |T|^2. \quad (3-14)$$

This is the case that we are interested in, as we will investigate the transmission property at the linear nondispersive–nonlinear dispersive interface. In this case the nonlinearity makes the cutoff frequency higher, and the effective cutoff frequency may be found by setting the left-hand side of expression (3-12) to zero, as

$$\omega_c' = \frac{\omega_c}{\left( 1 - (1/2) \beta_3 |A_i|^2 |T|^2 \right)^{1/2}} > \omega_c. \quad (3-15)$$

### 3.2.2 Solution in the Nonpropagation Case

As mentioned in Section 3.3.1, when expression (3-12) is satisfied at  $z=0$  there is no propagation, at least through a certain distance. The wave amplitude will decay until it reaches the threshold for transmission, as defined by expression (3-12) with the inequality replaced by an equality sign.

Now let us try to obtain a solution in the nonlinear dispersive medium in the nonpropagation case. We use the expression for  $k_t$  as in Eq. (3-13), and we rewrite it as

$$\begin{aligned} k_t(z) &= -j\tilde{k}_0 \left[ 1 - \left( \frac{\omega_c}{\omega_0} \right)^2 - \frac{1}{2} \beta_3 |A_t(z)|^2 \right]^{1/2} \\ &= -j\tilde{k}_0 \left\{ \frac{1}{2} \beta_3 |A_t(z)|^2 - \left[ 1 - \left( \frac{\omega_c}{\omega_0} \right)^2 \right] \right\}^{1/2} \end{aligned} \quad (3-16)$$

where the expression in  $\{ \}$  is a positive real function of  $A_t(z)$ , the amplitude at a distance  $z (> 0)$  from the interface (input plane) in the nonlinear dispersive region.

We define

$$\alpha \equiv 1 - \left( \frac{\omega_c}{\omega_0} \right)^2, \quad \beta \equiv \frac{1}{2} \beta_3. \quad (3-17)$$

It can be shown explicitly that both  $\alpha$  and  $\beta$  have to be positive to achieve bistability (see Cao and Banerjee, 1989). The differential equation for  $A_t$  becomes

$$\frac{dA_t(z)}{dz} = -\tilde{k}_0 \left[ \beta |A_t(z)|^2 - \alpha \right]^{1/2} A_t(z) \quad (3-18)$$

Without getting into details, we now give the solution for this differential equation. This reads

$$A_t(z) = \frac{(\alpha/\beta)^{1/2}}{\sin \sqrt{\alpha} \left\{ \tilde{k}_0 z + \frac{1}{\sqrt{\alpha}} \sin^{-1} \left[ (\alpha/\beta)^{1/2} / A_{t0} \right] \right\}} \quad (3-19)$$

where  $A_{t0} = |A_t(z=0)|$ . The above solution is valid up to a distance.

$$z = z_0 = \left\{ \frac{\pi}{2} - \sin^{-1} \left[ (\alpha/\beta)^{1/2} A_{t0} \right] \right\} / \alpha^{1/2} \tilde{k}_0 \quad (3-20)$$

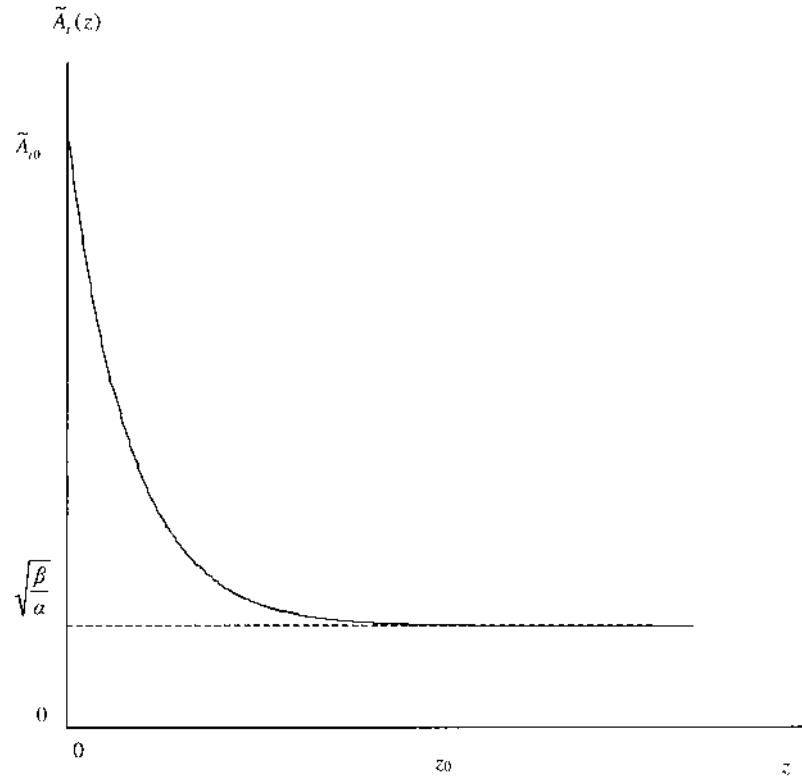
in the nonlinear medium.

It can be checked that

$$\psi(z, t) = (\alpha/\beta)^{1/2} \exp(j\phi_0) \exp(j\omega_0 t) \quad (3-21)$$

is a solution of Eq. (3-8) for  $z > z_0$ . Eqs. (3-19) and (3-21) complete the solution in the nonlinear dispersive medium, which is sketched in Fig. 10. This figure shows that if the incident wave amplitude is greater than the critical value  $(\alpha/\beta)^{1/2}$ , it first decays (nonexponentially), as in Eq. (3-19), and when it reaches the critical value  $(\alpha/\beta)^{1/2}$  it assumes a constant amplitude.

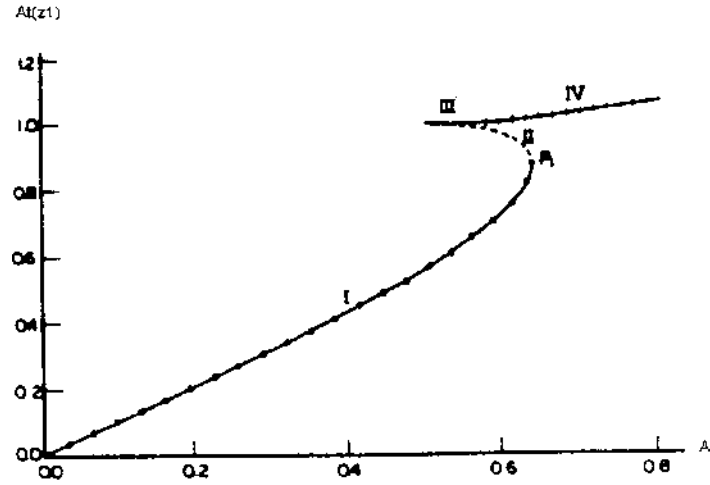
On the basis of the above solutions, we may now discuss the dependence of the transmitted wave amplitudes on the incident wave amplitudes. Let  $A_i$  denote the incident wave amplitude at the input. Assume that the output (or observation) plane is located at  $z_1$ , a fixed distance to the right of the interface. The amplitude of the transmitted wave in the output plane is denoted by  $A_t(z_1)$ .  $z_1$  may be larger or smaller than  $z_0$ , which is defined as the position at which  $A_t$  attains the critical value  $(\alpha/\beta)^{1/2}$ .  $z_0$  is a function of  $A_{t0}$ , the transmitted wave amplitude at  $z = 0$ , and hence is a function of the incident wave amplitude  $A_i$  through the boundary conditions. The stronger  $A_i$  is the larger  $z_0$  will be. [It takes a longer distance for a stronger incident wave to decay to the critical value  $(\alpha/\beta)^{1/2}$ .]



**Figure 10** Nature of the nonpropagating solution in the nonlinear dispersive medium. The fall-off is nonexponential, and the solution becomes a constant at  $z = z_0$ .

The relation between  $A_t(z_1)$  and  $A_i$  is shown in Fig. 11 for a set of suitably chosen parameters  $\alpha$ ,  $\beta$ , and  $z_1$ . The curve in Fig. 11 comprises two main parts: the lower part (I and II), which is the solution for  $A_{t0} < (\alpha/\beta)^{1/2}$  (transmission), and the upper part (III and IV) for  $A_{t0} < (\alpha/\beta)^{1/2}$  (non-propagation), with III  $A_{t0} < A_{t0}'$  and IV for  $A_{t0} > A_{t0}'$ ; where  $A_{t0} > A_{t0}'$ ; where  $A_{t0}' = (\alpha/\beta)^{1/2} / \sin(\frac{\pi}{2} - \alpha^{1/2} \tilde{k}_0 z_1)$ .

Fig. 11 gives an S-shaped input-output curve, which is typically associated with the hysteresis and bistability property of a system. Note that there is a certain region where there are multiple values of  $A_t(z_1)$  corresponding to a single value of  $A_i$ . This multiple-valued region is a necessary condition for ensuring hysteresis and bistability. It can be argued that the two param-



**Figure 11** The S-shaped input-output steady state curve for a linear nondispersive/nonlinear dispersive interface [Cao and Banerjee (1989)].

eters  $\alpha$  and  $\beta$  must be positive in order for there to be a multiple-valued region, which requires some section(s) with negative slope(s) on the  $A_t(z_1)$  vs.  $A_i$  curve.

#### 4 CONCLUSION

In this chapter, we have discussed different optically bistable systems, viz., a unidirectional ring cavity in the absorptive and dispersive regimes, a transverse optically bistable system, and bistability stemming from light transmission and reflection at a linear–nonlinear interface. The transient response in each case and stability analysis have been left out for the sake of simplicity. Various other bistable and switching devices have been researched and proposed, notably in other Kerr media and in semiconductors, thermal bistability, SEED devices, etc. Two-photon optical bistability has been demonstrated by various researchers, see for instance, Giacotino et al. (1980). Optical bistability during phase conjugation has been demonstrated by Winful and Marburger (1980), and using photonfraction materials, by Kukhtarev and Semenets (1981). For an excellent treatment, see Gibbs (1985). For practical applications, devices should be small and fast, and operate at low powers. When these devices will find their place in realizable optical computers that can challenge their electronic counterparts still remains to be seen.

## 5 PROBLEMS

1. For absorptive bistability, the steady state can be written as

$$I_i = I_t \left[ 1 + \frac{2C}{1 + I_t} \right]^2$$

where  $C$  is a constant. Determine the minimum value of  $C$  for bistability to occur.

2. Rederive the absorptive bistability steady state in the case then the cavity has a finite loss  $a = 1 - r - t$  in each coating and an unsaturable absorption coefficient (intensity) of  $\alpha_B$ . Is bistability more difficult to obtain?
3. How are the incident and transmitted phases related for the cases of absorptive and dispersive bistability?
4. A qualitative argument for stability of the ring cavity shows that all steady states where the slope of the input–output curve is negative are unstable. Show that this is, in general, true for any bistable system with

$$\frac{I_t}{I_i} = \mathfrak{S}(\beta)$$

where  $\beta$  evolves according to

$$\tau \frac{d\beta}{dt} = -(\beta - \beta_1 - \beta_3 I_t),$$

with  $\beta_1, \beta_3$  being constants, and  $\tau$  is a characteristic time constant.

5. For a linear–nonlinear interface and for normal incidence, find the reflection and transmission coefficients across the interface. Plot the transmitted and reflected intensities as functions of the input intensity for  $n_2 > 0$  and  $n_2 < 0$ .
6. Repeat Problem 5 for “grazing” incidence.

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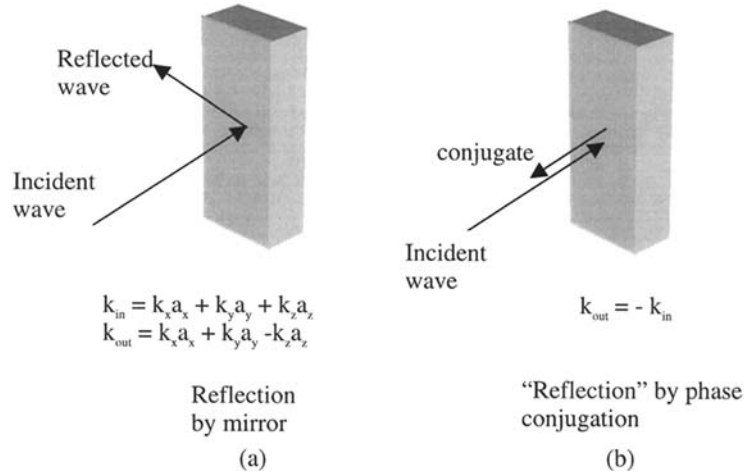
## Optical Phase Conjugation

In Chaps. 4 and 5, we have discussed some consequences of optical propagation in a cubically nonlinear medium, viz., self-refraction and optical bistability. In this chapter, we will study yet another consequence of wave interaction in a cubically nonlinear medium—*optical phase conjugation and four-wave mixing*.

*Optical phase conjugation* is a technique that employs the cubic nonlinearity of the optical medium to precisely reverse the overall phase factor, and sometimes the direction, of an arbitrary beam of light. It was first discovered in the early 1970s by Zeldovich et al (1972) using *stimulated Brillouin scattering* (SBS), which is discussed in Chap. 7. We can, in some sense, regard the process as reflection of light from a mirror with unusual image-transformation properties. To illustrate this point, note that a conventional mirror [Fig. 1(a)] changes the sign of the  $\mathbf{k}$  vector component normal to the mirror surface while leaving the tangential component unchanged. On the other hand, phase conjugation [Fig. 1(b)] causes an inversion of the entire vector  $\mathbf{k}$ , so that the incident ray exactly returns upon itself [Yariv, 1997]. It is not hard to argue that an incident converging (diverging) beam would be *conjugated* into a diverging (converging) beam. Mathematically speaking, this means that if  $\psi_p$  represents the incident phasor, then  $\psi_p^*$  would represent the phase-conjugated phasor.

Basic to the understanding of the principles underlying the rapidly developing field of phase conjugate optics are certain properties of a wave (known as the conjugate wave) that is generated from a given input wave by reversing the latter's phase at every point on its wave front. The phase reversal can be achieved in a variety of ways, including SBS three-wave mixing in a crystal, or the so-called *degenerate four-wave mixing* (DFWM).

In the standard scheme of degenerate four-wave mixing, the conjugated wave with a reversed wavefront is generated by the interaction between the *probe* wave and the counterpropagating *pump* waves (Hellwarth, 1977, Yariv and Pepper, 1977). The term *pump* refers to a strong beam of light that is often



**Figure 1** Principle of phase conjugation illustrated through a comparison between (a) conventional reflection from a mirror and (b) reflection by phase conjugation.

assumed to be undepleted in energy during the interaction process. The *probe* comprises a weaker beam or pulse of light, of which the conjugate is desired. Phase conjugation is often achieved by having counterpropagating pump waves in a Kerr-type optically nonlinear medium, into which the probe is also introduced. Without getting into the details of the nonlinear interaction for the moment, let us state that this process is readily visualized in  $\mathbf{k}$  space using a simple geometrical construction. Addition of the ray-tracing technique (not discussed here) provides more insight into this process, including determination of the image location. We would like to stress the fact that the  $\mathbf{k}$  space formulation cannot give us the information of the location of the image; rather it can only predict the nature of the image. The reason for this is that our  $\mathbf{k}$  space formalism, in a loose sense, only contains information about the direction of the “rays” and not their actual location. On the other hand, the ray diagram approach predicts the location of the image, but cannot predict if or not the conjugate is pulsed. In fact, one may view both these procedures as, to some extent, complimentary, and really subsets of a rigorous wave treatment to be discussed later.

## 1 COMPARISON WITH HOLOGRAPHY

At this time, it is instructive to refresh our memory with the basic principles of *wavefront reconstruction* imaging or *holography* [Poon and Ba-

nerjee (2001)]. Consider two wavefronts, represented by their respective envelopes,

$$\begin{aligned}\psi_{e1} &= a_1 \exp(-j\phi_1) \\ \psi_{e2} &= a_2 \exp(-j\phi_2),\end{aligned}\tag{1-1}$$

to interfere in space, resulting in an intensity

$$\begin{aligned}I &= |\psi_{e1} + \psi_{e2}|^2 = |\psi_{e1}|^2 + |\psi_{e2}|^2 + \psi_{e1}\psi_{e2}^* + \psi_{e1}^*\psi_{e2} \\ &= a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2).\end{aligned}\tag{1-2}$$

The intensity distribution is then *recorded* on a film, and the film is developed to generate a hologram with a transparency function that is proportional to  $I$ . To understand the *reconstruction* process, assume that the hologram is now illuminated with a reconstruction wave  $\psi_{e3}$ . The field behind the transparency is then proportional to

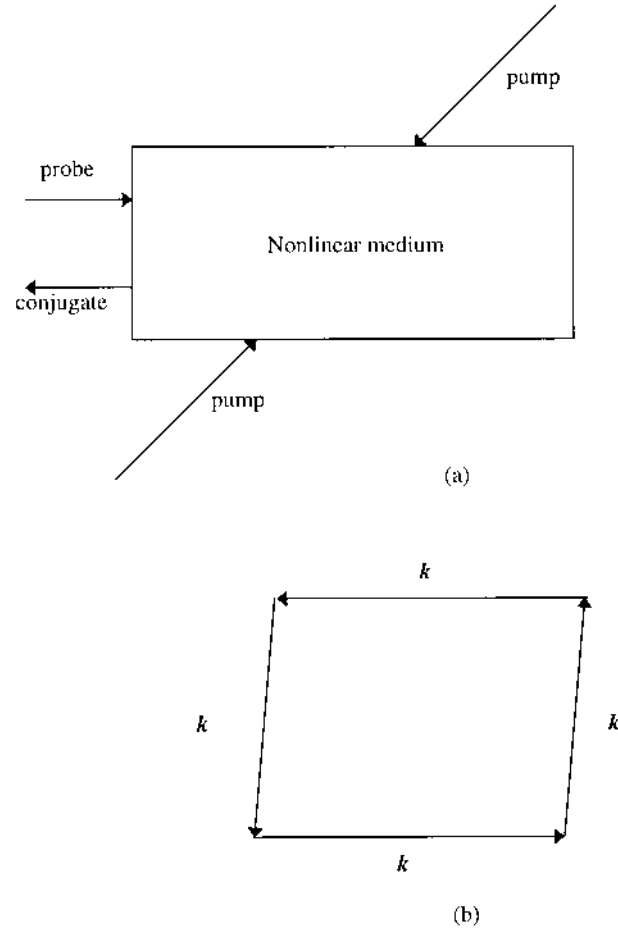
$$\psi_{e3} \left[ |\psi_{e1}|^2 + |\psi_{e2}|^2 + \psi_{e1}\psi_{e2}^* + \psi_{e1}^*\psi_{e2} \right].\tag{1-3}$$

Note that if  $\psi_{e3}$  is an exact duplication of one of the original waves, namely,  $\psi_{e2}$ , which we can call the *reference wave*, the third term in Eq. (1-3) becomes equal to  $\psi_{e1}|\psi_{e2}|^2$  and, hence, is a duplicate of  $\psi_{e1}$ , which we can call the *object wave*. However, if  $\psi_{e3}$  becomes  $\psi_{e2}^*$ , observe that the fourth term of Eq. (1-3) (which we shall name  $\psi_{e4}$ ) is proportional to  $\psi_{e1}^*$  and hence represents the conjugate of the object wavefront.

From the discussion in the previous paragraph, it is clear that we can achieve phase conjugation using holographic techniques, although in the conventional sense, the process is slow and not in real time, owing to the efforts in making the hologram. Note that in the preceding discussion, we tacitly talked in terms of envelopes rather than phasors because reversal of the entire  $\mathbf{k}$  vector is not possible using holographic techniques. The process of introducing  $\psi_{p1}$ ,  $\psi_{p2}$ , and  $\psi_{p3}$  all at once in a cubically nonlinear medium to generate  $\psi_{p4}$  speeds up the conjugation and is referred to as the *four-wave mixing problem*. The entire  $\mathbf{k}$  vector of  $\psi_{p4}$  is the negative of that of  $\psi_{p1}$ . We will discuss this at some length later in the chapter.

## 2 THE $\mathbf{k}$ -SPACE FORMALISM

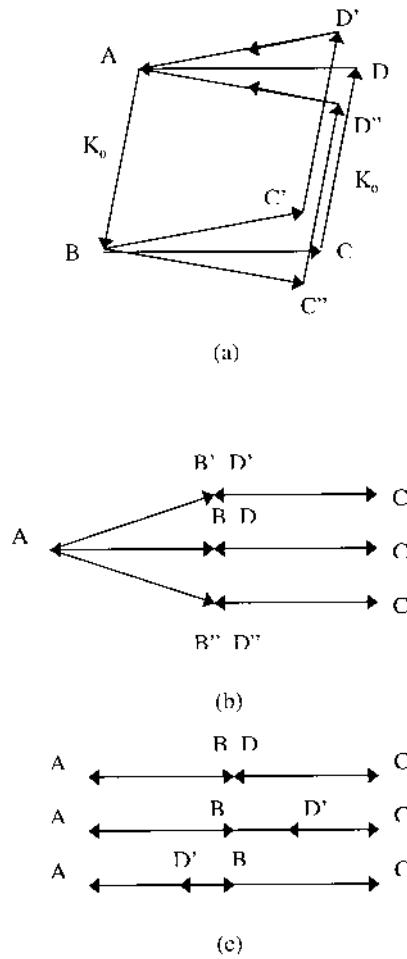
In the standard scheme of DFWM, the conjugated wave with a reversed wavefront is generated by the interaction between the probe and the counterpropagating pumps. Fig. 2(a) shows the conventional configuration



**Figure 2** Phase conjugation with noncollinear geometry. (a) The conventional configuration, and (b)  $k$ -space representation.

for phase conjugation for noncollinear interaction. Fig. 2(b) depicts the  $k$  space representation of the same system. The length of the side of the parallelogram is  $k_0$ . Absence of dispersion ensures that if all participating frequencies are equal ( $=\omega_0$ ), so are the corresponding propagation constants ( $k_0$ ), with  $\omega_0/k_0 = \text{constant}$ . Thus if one or more of the participating frequencies be  $\omega_0 + \Delta\omega$ , the corresponding propagation constant(s) should be  $k_0 + \Delta k = (\omega_0 + \Delta\omega)/(\omega_0/k_0)$ . The conservation of energy and momentum describes a parametric interaction of the form  $\omega_0 + \omega_0 - \omega_0 \rightarrow \omega_0$  (in frequency space) and  $k_0 + k_0 - k_0 \rightarrow k_0$  (in  $k$  space).

To understand the effect of phase conjugation on a probe having an amplitude and/or phase profile, consider the parallelogram ABCD shown in Fig. 3(a), with edge lengths equal to  $k_0$ , which represents four-wave mixing between the probe (BC), the pumps (AB, CD), and the conjugate (DA), when all the participants are plane waves with propagation constant  $k_0$ . If the probe has an amplitude and/or phase profile, it can be visualized in terms of its angular plane wave spectrum. Thus in Fig. 3(a)  $BC''$ , BC denote representa-



**Figure 3** (a)  $k$ -space picture of noncollinear four-wave mixing where the probe may have an amplitude/phase profile; (b) same as (a) but for a nominally collinear geometry; (c)  $k$ -space picture of collinear four-wave mixing when a pump is pulsed.

tive  $\mathbf{k}$  vectors of the angular plane wave spectrum, with lengths once again equal to  $k_0$ , under the paraxial approximation. By the term “paraxial approximation,” we only mean that the components of the  $\mathbf{k}$  vectors transverse to the direction of nominal propagation are small compared to  $k_0$ . As our construction shows,  $\mathbf{D}'\mathbf{A}$  and  $\mathbf{D}''\mathbf{A}$  are the resultant  $\mathbf{k}$  vectors of the angular plane wave spectrum of the conjugate. Considering the  $\mathbf{k}$  vectors to be akin to rays, it is easy to argue that the *diverging* probe wave results in a *converging* “conjugate,” justifying its name. While Fig. 3(a) represents DFWM in a noncollinear geometry, it collapses to Fig. 3(b) for nominally collinear geometry. By “nominally,” we mean to stress that if all participants of the DFWM are single plane waves, the vector diagram is collinear, as shown by the “collapsed” parallelogram ABCD. Note that in this case, there is no distinction between the pumps, and the probe and conjugate. To more succinctly bring out this effect to the readers, we have therefore tacitly taken AB to be the probe and DA to be the conjugate. Thus  $\mathbf{AB}'$ ,  $\mathbf{AB}''$  now denote the angular plane wave spectrum of the probe; and  $\mathbf{D}'\mathbf{A}$ ,  $\mathbf{D}''\mathbf{A}$  that of the conjugate. It is clear that in the presence of an amplitude/phase profile of the probe, the interaction loses its collinearity. Thus the  $\mathbf{k}$  space formalism helps in conveying the physical picture in a simple and straightforward way. For details, readers are referred to Ghosh and Banerjee (1993).

If, now, one of the participants is pulsed instead of having an amplitude/phase profile, DFWM may be similarly studied in  $\mathbf{k}$  space. A pulsed wave has a frequency profile that is represented here as a spread in the magnitudes of the propagation vectors associated with the pulsed wave. As shown in Fig. 3(c) for collinear interaction, the  $\mathbf{k}$  spectrum of the pulsed pump contains collinear  $\mathbf{k}$  vectors of different lengths conforming to the frequency spectrum of the pulse (see lengths CD,  $\mathbf{CD}'$ ,  $\mathbf{CD}''$ ). The conjugate  $\mathbf{k}$  vectors correspondingly become DA,  $\mathbf{D}'\mathbf{A}$ , and  $\mathbf{D}''\mathbf{A}$ , suggesting coupling from the upper side band (USB) [lower side band (LSB)] of the pump to the LSB (USB) of the conjugate.

It is evident from the above discussion that there is information transfer from the probe to the conjugate if the probe has an amplitude and/or phase profile. Similarly, there is information transfer from the pump to the conjugate if one of the pumps is pulsed. In what follows, we will illustrate the use of our simple  $\mathbf{k}$  space formalism in investigating the nature of the conjugate for different probe and pump characteristics and in collinear and noncollinear geometries, which are combinations of the cases shown in Fig. 3.

## 2.1 Collinear Geometry: Probe with Amplitude and/or Phase Profile and Pulsed Pump

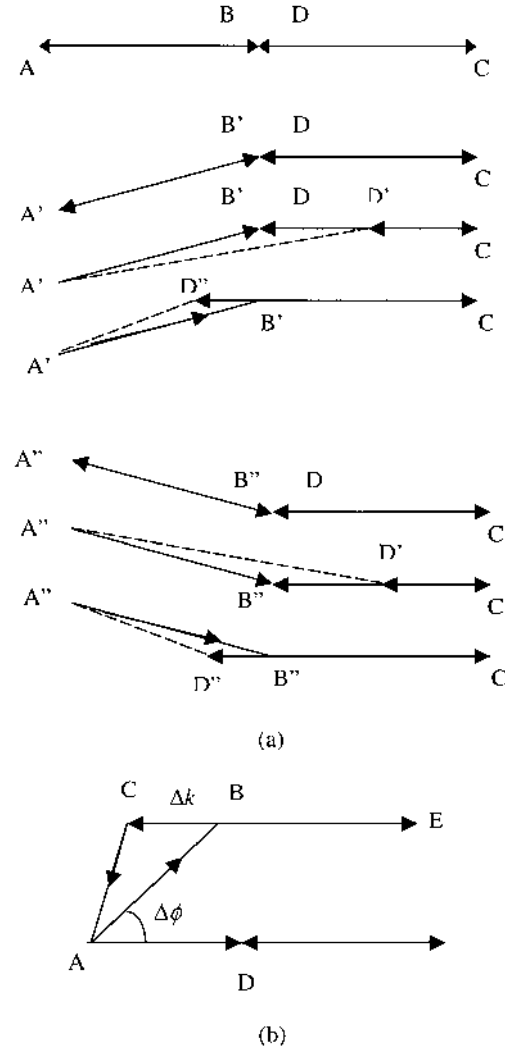
We consider DFWM in a collinear geometry where the probe has an amplitude and/or phase profile and where one of the pumps is pulsed. This case

has been theoretically investigated by Bakos et al. (1989). Their analytical results can be readily duplicated using our simple  $\mathbf{k}$  space formalism, as discussed below. Fig. 4(a) shows the possible four-wave interactions. Here we have represented the probe by lines AB, A'B', and A''B'' to accommodate its amplitude and/or phase profile. Similarly, we have represented the backward traveling pump by lines CD, CD', and CD'' to illustrate its pulsed nature. From the figure, line segments DA, DA', D'A', D''A', D'A'', and D''A'' represent all possible contributions to the conjugate. As is evident from the figure, the conjugate will contain the amplitude and/or phase profile in addition to being pulsed at the same time. This is because the constituent  $\mathbf{k}$  vectors of the conjugate have different directions and lengths. To quantify our physical arguments, we use simple geometry to predict the nature of the conjugate in terms of the characteristics of the probe and the pump. To this end, we refer to Fig. 4(b), which depicts a typical situation where the probe has an amplitude and/or phase profile (represented by  $\Delta\phi$ ) and where the pump is pulsed (represented by  $\Delta k$ ). CA represents the resulting conjugate and BE is the forward traveling pump. From geometry,  $|AC| = k_0 - \Delta k + \text{higher order terms}$ , where  $k_0$  is the nominal propagation constant for all four interacting waves. Thus  $(\Delta k)_c \equiv |AC| - k_0 \equiv -(\Delta k)$ , and denotes the change in the length of the  $\mathbf{k}$  vector of the conjugate, implying a pulsed output. Also,  $\angle CAD = \Delta\phi_c = \Delta\phi + \text{higher-order terms}$ . The neglect of higher-order terms in the above calculations assumes that  $\Delta k \ll k_0$  and that  $\Delta\phi$  is small. The above results show that the conjugate will have wavefronts that are approximately conjugated to the wavefronts of the probe, and the temporal spectrum of the conjugate  $[\alpha(\Delta k)_c]$  is of the same shape as the temporal spectrum of the pump. The negative sign in the relation  $(\Delta k)_c \equiv -(\Delta k)$  indicates coupling from the upper sideband (USB) of the pump to the lower sideband (LSB) of the conjugate, and vice versa. This, in turn, implies that the conjugate is approximately of the same duration as the pulsed pump. These findings are identical to the results in Bakos et al. (1989).

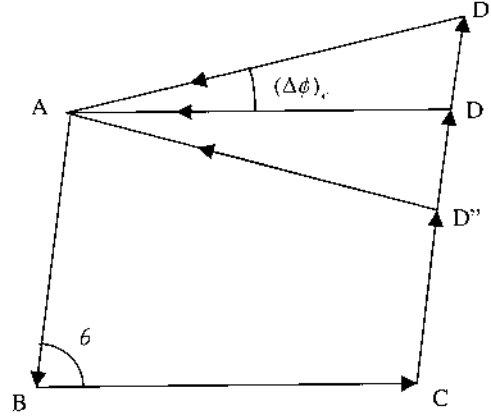
## 2.2 Noncollinear Geometry

### 2.2.1 Pulsed Pump

In a similar fashion, we can study the nature of the conjugate during DFWM in a noncollinear geometry if one of the pump is pulsed. This is shown in Fig. 5. The pulsed pump is represented by the lines of CD, CD', and CD''. The corresponding conjugates are represented by DA, D'A, and D''A, respectively. It is clear that the resulting conjugate is pulsed and has an amplitude and/or phase profile. Referring to Fig. 5, it can be shown that  $(\Delta k)_c \equiv (\Delta k)\cos\theta$ , where  $(\Delta k)_c = AD' - AD$ , and  $(\Delta\phi)_c \equiv (\Delta\phi/k_0)\sin\theta$ . This shows that the conjugate pulse duration  $\tau_c$  is expected to be of the order of  $\tau_p/\cos\theta$ , where



**Figure 4** (a) Possible four-wave interactions in nominally collinear geometry when the probe has an amplitude/phase profile and the backward traveling pump is pulsed; (b) typical geometry to calculate  $(\Delta k)_c$  and  $(\Delta \phi)_c$ . The lengths AB, BE, and AD and each equal to  $k_0$ .

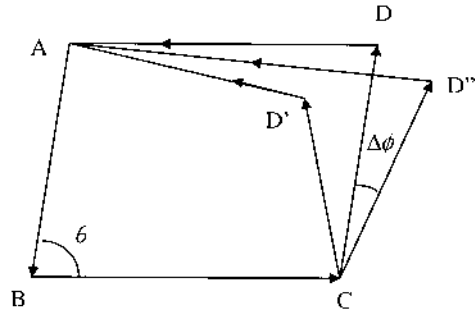


**Figure 5** Typical geometry to analyze the nature of the conjugate during DFWM in a noncollinear geometry where one of the pump is pulsed. The lengths of AB, BC, CD and DA are each equal to  $k_0$ , while the lengths of DD', DD'' are  $\Delta k$ .

$\tau_p$  is the duration of the pulsed pump. Note that the conjugate light is spatially modulated in accordance with the temporal modulation of the pump. Furthermore, the amount of spatial modulation can be change by varying  $\theta$ .

### 2.2.2 Pump with Amplitude and/or Phase Profile

If the pump has an amplitude and/or phase profile, the corresponding  $k$  space representation of DFWM in a noncollinear geometry is shown in Fig. 6. Once again, CD, CD', CD'' represent the pump, with DA, D'A, and D''A depicting the respective conjugate waves. In this case,  $(\Delta k)_c = D''A - DA \simeq k_0(\Delta\phi)\sin\theta$ ;



**Figure 6** Typical geometry to analyze the nature of the conjugate during DFWM in a noncollinear geometry where one of the pumps has an amplitude/phase profile. The lengths AB, BC, CD, CD', CD'' and DA are each equal to  $k_0$ .

and  $(\Delta\phi)_c = \langle DAD \equiv 0$ . This means that the conjugate will be predominantly pulsed, with an almost uniform amplitude and/or phase profile. In this case, the information in the spatial modulation of the pump is primarily transferred as pulsed information in the conjugate. The  $\mathbf{k}$  space representation, as shown in Fig. 6, is qualitatively unchanged even if the pump is pulsed. It is to be noted that many more special cases (corresponding to choices of temporal and spatial characteristics of the probe and pumps, and the choice of geometries) are all straightforward applications of the method presented above.

### 3 SEMICLASSICAL ANALYSIS: CW CASE

We start from Eq. (3-9) of Chap. 2 with  $\beta_2 = 0$ ,

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \nabla^2 \psi = \frac{2\beta_3}{3} \frac{\partial^2 \psi^3}{\partial t^2}, \quad (3-1)$$

where  $\beta_3$  is proportional to  $-n_2$  and  $-\chi^{(3)}$  as shown in Eqs. (1-6) and (1-7) of Chap. 4. Writing

$$\psi(x, y, z, t) = \text{Re}[\psi_p(x, y, z, t)\exp(j\omega_0 t)]$$

in Eq. (3-1), we obtain

$$2j\omega_0 \frac{\partial \psi_p}{\partial t} - \omega_0^2 \psi_p - v^2 \nabla^2 \psi_p = -\frac{1}{2} \beta_3 \omega_0^2 |\psi_p|^2 \psi_p \quad (3-2)$$

where we have restricted ourselves around the carrier frequency  $\omega_0$  and have assumed  $\psi_p$  to be a slowly varying function of time in the sense that

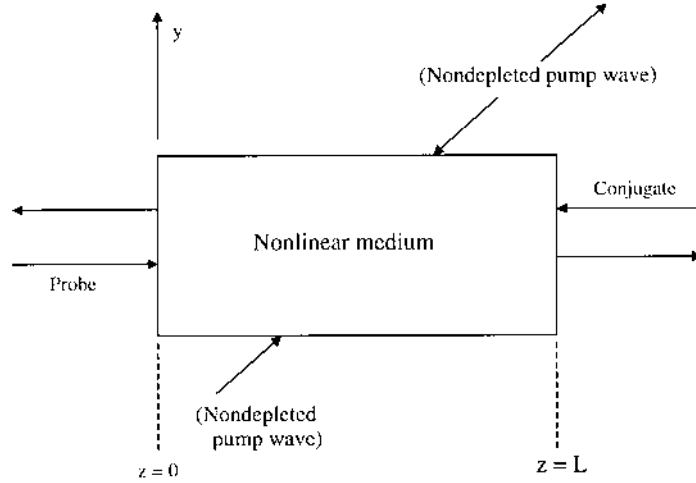
$$\left| \frac{\partial^2 \psi_p}{\partial t^2} \right| \ll \omega_0 \left| \frac{\partial \psi_p}{\partial t} \right|.$$

Now consider the geometry of Fig. 7. We assume the total phasor  $\psi_p$  in the interaction region (nonlinear medium) to be the sum of four phasors as

$$\psi_p = \bar{\epsilon} \psi_{p1} + \psi_{p2} + \psi_{p3} + \bar{\epsilon} \psi_{p4} \quad (3-3a)$$

$$= \bar{\epsilon} \psi_{e1}(t) \exp(-jk_0 z) + \psi_{e2}(t) \exp[-j(k_{y0} y + k_{z0} z)] \\ + \psi_{e3}(t) \exp[j(k_{y0} y + k_{z0} z)] + \bar{\epsilon} \psi_{e4}(t) \exp(jk_0 z). \quad (3-3b)$$

In Eq. (3-3a),  $\psi_{p2}$  and  $\psi_{p3}$  represent the phasors corresponding to the *pump* waves, with  $\psi_{p1}$  and  $\psi_{p4}$  representing the *probe* and the *conjugate* wave phasors, respectively. The quantity  $\bar{\epsilon}$  is a bookkeeping parameter that is taken to be small to emphasize that the pump energy is much larger than the



**Figure 7** Basic geometry of phase conjugation by four-wave mixing.

probe and conjugate wave energies. Equation (3-3b) specifies the direction of propagation of the components of the phasor  $\psi_p$ . The pumps are contra-directed and travel in directions, specified by the direction cosines,  $\pm(k_{y0}, k_{z0})$ , respectively, where

$$k_{y0}^2 + k_{z0}^2 = \bar{k}_0^2. \quad (3-4)$$

The probe and the conjugate travel in the  $\pm z$  directions, respectively. Note that we have purposely assumed the propagation constant for the pumps to be different from  $k_0$ , the propagation constant of the probe and conjugate. This is because in a nonlinear medium, the *effective propagation constant* (and hence the effective phase velocity) depends on the amplitudes of the waves propagating in the medium. Because the amplitudes of the pumps are much larger than those of the probe and the conjugate, we expect the effective propagation constant to be different in the direction of the pumps, but not in the direction of the probe and conjugate.

Our aim is to study the evolution of the probe and the conjugate waves, as well as to understand the behavior of the pumps. Because all the participating waves propagate in different directions, we find it convenient to track their amplitudes as functions of a common variable. Therefore we choose this variable to be  $t$ , which has the dimension of time. To be precise,  $t$  represents the time each participating wave travels in its specified direction. This is a little different from our approach during the study of second harmonic generation

(see Chap. 3), where we tracked each spectral component as a function of  $z$ , because all spectral components traveled in the same direction.

If we substitute Eq. (3-3b) into Eq. (3-2), the left-hand side yields the following coefficients for the four complex exponentials:

$$\begin{aligned}
 & \bar{\epsilon} 2j\omega_0 \frac{d\psi_{e1}}{dt}, & \text{for exp } (-jk_0z); \\
 & 2j\omega_0 \frac{d\psi_{e2}}{dt} - \left(\omega_0^2 - v^2 \bar{k}_0^2\right) \psi_{e2}, & \text{for exp } [-j(k_{y0}y + k_{z0}z)]; \\
 & 2j\omega_0 \frac{d\psi_{e3}}{dt} - \left(\omega_0^2 - v^2 \bar{k}_0^2\right) \psi_{e3}, & \text{for exp } [-j(k_{y0}y + k_{z0}z)]; \\
 & \bar{\epsilon} 2j\omega_0 \frac{d\psi_{e4}}{dt}, & \text{for exp } (jk_0z)
 \end{aligned} \tag{3-5}$$

where we have set  $\omega_0/k_0 = v$ . Evaluation of the right-hand side is more involved. We find  $|\psi_p|^2 \psi_p$  and only retain terms containing the exponentials listed in Eq. (3-5). Furthermore, for the exponentials of the form  $\exp(ik_0z)$  (pertaining to the probe and the conjugate), we only retain terms that involve  $\bar{\epsilon}$ , neglecting higher powers of  $\bar{\epsilon}$ . Similarly, for the exponentials of the form  $\exp[-j(k_{y0}y + k_{z0}z)]$  (pertaining to the contrapropagating pumps), we retain terms involving  $\bar{\epsilon}^0$  and neglect higher powers of  $\bar{\epsilon}$ . We list next the coefficients of the four exponentials in Eq. (3-5) that represent the contribution from  $|\psi_p|^2 \psi_p$ :

$$\begin{aligned}
 & \bar{\epsilon} \left[ 2\psi_{e1} \left( |\psi_{e2}|^2 + |\psi_{e3}|^2 \right) + 2\psi_{e2}\psi_{e3}\psi_{e4}^* \right] & \text{for exp } (-jk_0z); \\
 & \psi_{e2} \left( |\psi_{e2}|^2 + 2|\psi_{e3}|^2 \right), & \text{for exp } [(-jk_{y0}y + jk_{z0}z)]; \\
 & \psi_{e3} \left( |\psi_{e3}|^2 + 2|\psi_{e2}|^2 \right), & \text{for exp } [(jk_{y0}y + jk_{z0}z)]; \\
 & \bar{\epsilon} \left[ 2\psi_{e4} \left( |\psi_{e2}|^2 + |\psi_{e3}|^2 \right) + 2\psi_{e2}\psi_{e3}\psi_{e1}^* \right] & \text{for exp } (jk_0z).
 \end{aligned} \tag{3-6}$$

Incorporating Eq. (3-5) and (3-6) into Eq. (3-2), we obtain the following set of evolution equations for the probe, conjugate, and pump amplitudes:

$$\frac{d\psi_{e1}}{dt} = j \frac{\omega_0 \beta_3}{4} \left[ 2\psi_{e1} \left( |\psi_{e2}|^2 + |\psi_{e3}|^2 \right) + 2\psi_{e2}\psi_{e3}\psi_{e4}^* \right], \tag{3-7a}$$

$$\frac{d\psi_{e2}}{dt} = j \frac{\omega_0 \beta_3}{4} \left[ \psi_{e2} \left( |\psi_{e2}|^2 + 2|\psi_{e3}|^2 \right) \right] - \frac{j \left( \omega_0^2 - v^2 \bar{k}_0^2 \right)}{2\omega_0}, \tag{3-7b}$$

$$\frac{d\psi_{e3}}{dt} = j \frac{\omega_0 \beta_3}{4} \left[ \psi_{e3} \left( |\psi_{e3}|^2 + 2|\psi_{e2}|^2 \right) \right] - \frac{j \left( \omega_0^2 - v^2 \bar{k}_0^2 \right)}{2\omega_0}, \tag{3-7c}$$

$$\frac{d\psi_{e4}}{dt} = j \frac{\omega_0 \beta_3}{4} \left[ 2\psi_{e4} \left( |\psi_{e2}|^2 + |\psi_{e3}|^2 \right) + 2\psi_{e2}\psi_{e3}\psi_{e1}^* \right]. \quad (3-7d)$$

Assuming that the pumps are undepleted, we can set  $d\psi_{e2}/dt = d\psi_{e3}/dt = 0$ . Then writing

$$|\psi_{e2}| = |\psi_{e3}| = A, \quad (3-8)$$

we can evaluate the modified propagation constant  $\bar{k}_0$  in the direction of the pumps. From Eq. (3-7b) or (3-7c), this becomes

$$\bar{k}_0 = k_0 \left[ 1 - \frac{3}{4} \beta_3 A^2 \right]. \quad (3-9)$$

Hence the field profile along the direction of the pumps becomes

$$\begin{aligned} & \text{Re} \left[ A \exp(j\omega_0 t) \exp(-j\bar{k}_0 z') + A \exp(j\omega_0 t) \exp(j\bar{k}_0 z') \right] \\ &= 2A \cos(\omega_0 t) \cos(\bar{k}_0 z'), \end{aligned} \quad (3-10)$$

where  $z'$  denotes the direction of the pumps and where  $\bar{k}_0$  is given by Eq. (3-9). Hence, in the direction of the pumps, we observe a standing wave with a spatial period equal to  $2\pi/\bar{k}_0$ . We can visualize this standing wave to form a phase grating, which scatters the probe field and generates the conjugate. From Eqs. (3-7a) and (3-7d) with Eq. (3-8), we can track the evolution of the probe and the conjugate by means of the coupled ODEs

$$\frac{d\psi_{e1}}{dt} = j \frac{\omega_0 \beta_3 A^2}{2} \left[ 2\psi_{e1} + \psi_{e4}^* \right], \quad (3-11a)$$

$$\frac{d\psi_{e4}^*}{dt} = j \frac{\omega_0 \beta_3 A^2}{2} \left[ \psi_{e1} + 2\psi_{e4}^* \right], \quad (3-11b)$$

where, in writing Eq. (3-11b), we have taken the complex conjugate of Eq. (3-7d). Recall that the variable  $t$  in Eqs. (3-11a) and (3-11b) is representative of the distance traveled by the probe and the conjugate in their respective directions of propagation  $z$  and  $-z$ , respectively. Thus we can recast Eqs. (3-11a) and (3-11b) to ODEs with  $z$  as the independent variable provided we replace  $t$  by  $z/v$  in Eq. (3-11a) and by  $-z/v$  in Eq. (3-11b). With  $\omega_0 = vk_0$ , this gives

$$\frac{d\psi_{e1}}{dz} = j \frac{k_0 \beta_3 A^2}{2} \left[ 2\psi_{e1} + \psi_{e4}^* \right] \quad (3-12a)$$

$$\frac{d\psi_{e4}^*}{dz} = j \frac{k_0 \beta_3 A^2}{2} \left[ \psi_{e1} + 2\psi_{e4}^* \right]. \quad (3-12b)$$

To solve this system of equations, we introduce

$$\tilde{\psi}_{e1} = \psi_{e1} \exp[-j\beta_3 k_0 A^2 z], \quad \tilde{\psi}_{e4} = \psi_{e4} \exp[j\beta_3 k_0 A^2 z] \quad (3-13)$$

to recast Eqs. (3-12a) and (3-12b) in the form

$$\frac{d\tilde{\psi}_{e1}}{dz} = j\tilde{\delta}\psi_{e4}^* \quad \frac{d\tilde{\psi}_{e4}^*}{dz} = -j\tilde{\delta}\tilde{\psi}_{e1} \quad (3-14)$$

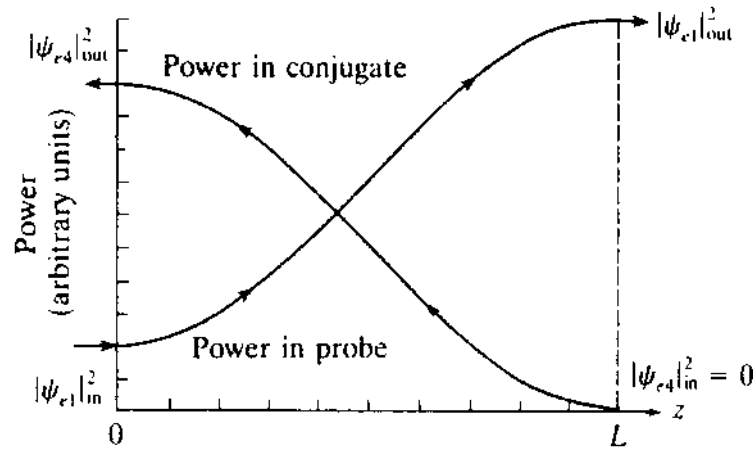
where

$$\tilde{\delta} = \frac{k_0 \beta_3 A^2}{2}. \quad (3-15)$$

The solutions to Eq. (3-14) for initial conditions  $\tilde{\psi}_{e1}(0)$  and  $\tilde{\psi}_{e4}(L)$  are given by [Fisher (1983), Yariv (1997)]

$$\tilde{\psi}_{e1}(z) = j \left[ \frac{|\tilde{\delta}| \cos(|\tilde{\delta}|z)}{\tilde{\delta} \cos(|\tilde{\delta}|L)} \right] \psi_{e4}^*(L) + \left[ \frac{\cos(|\tilde{\delta}|z - L)}{\cos(|\tilde{\delta}|L)} \right] \tilde{\psi}_{e1}(0) \quad (3-16a)$$

$$\tilde{\psi}_{e4}(z) = \left[ \frac{\cos(|\tilde{\delta}|z)}{\cos(|\tilde{\delta}|L)} \right] \tilde{\psi}_{e4}(L) + j \left[ \frac{|\tilde{\delta}| \cos(|\tilde{\delta}|z - L)}{\tilde{\delta} \cos(|\tilde{\delta}|L)} \right] \tilde{\psi}_{e1}^*(0). \quad (3-16b)$$



**Figure 8** Variation of probe and conjugate during propagation through a nonlinear phase-conjugating medium.

In the practical case of phase conjugation,  $\tilde{\psi}_{e1}(0)$  is finite, whereas  $\tilde{\psi}_{e4}(L) = 0$ . In this case, the nonlinearly reflected wave at the input plane ( $z=0$ ) is

$$\tilde{\psi}_{e4}(0) = -j \left[ \frac{\tilde{\delta}}{|\tilde{\delta}|} \tan(|\tilde{\delta}|L) \right] \tilde{\psi}_{e1}^*(0), \quad (3-17)$$

where  $\tilde{\delta}$  is defined in Eq. (3-15). Note that  $\tilde{\psi}_{e4}$  in Eq. (3-17) is proportional to the complex conjugate of  $\tilde{\psi}_{e1}$ , as expected. The variation of the probe and conjugate power as a function of distance is shown in Fig. 8.

#### 4 PHASE CONJUGATION OF PULSES AND TRANSIENT PHASE CONJUGATE RESPONSE

In the previous section, all the four interacting waves were a function of one parameter, viz.  $t$ . This was performed to simplify the calculations and alleviate the need for keeping track of the respective directions of propagation of the interacting waves. To include time and space dependence for the probe and the conjugate is, undoubtedly, more involved and entails writing down the explicit space dependence of the pumps as well. We are going to avoid the rigorous derivation but state the final equations for the space-time evolution of the probe and the conjugate, assuming undepleted pumps [Sudyam and Fisher (1983)]

$$\begin{aligned} \frac{\partial \tilde{\psi}_{e1}}{\partial z} + \frac{1}{v} \frac{\partial \tilde{\psi}_{e1}}{\partial t} &= j\tilde{\delta}\tilde{\psi}_{e4}^* \\ \frac{\partial \tilde{\psi}_{e4}^*}{\partial z} - \frac{1}{v} \frac{\partial \tilde{\psi}_{e4}^*}{\partial t} &= j\tilde{\delta}\tilde{\psi}_{e1}. \end{aligned} \quad (4-1)$$

Note that upon setting  $\partial/\partial t$  to zero, Eq. (4-1) revert to Eq. (3-14). The quantity  $v$  denotes the group velocity. Note that the first equation models a pulse traveling along  $+z$  (assuming  $v > 0$ ) whereas the second equation models a backward traveling conjugate pulse.

Equation (4-1) can be solved using transform methods, viz., taking Fourier transform in  $t$  and Laplace transform in  $z$ . Boundary conditions may be set as  $\tilde{\psi}_{e1}(0, t) = f(t)$ ,  $\tilde{\psi}_{e4}(L, t) = 0$  where  $L$  is the length of the nonlinear material. Taking Fourier transform *w.r.t.*  $t$ , we have

$$\begin{aligned} \frac{d\tilde{\Psi}_{e1}(z, \Omega)}{dz} + j\frac{\Omega}{v}\tilde{\Psi}_{e1} - j\tilde{\delta}\tilde{\Psi}_{e4}^* &= 0 \\ \frac{d\tilde{\Psi}_{e4}^*(z, \Omega)}{dz} - j\frac{\Omega}{v}\tilde{\Psi}_{e4}^* - j\tilde{\delta}\tilde{\Psi}_{e1} &= 0 \end{aligned} \quad (4-2)$$

where  $\Omega$  is the temporal frequency and  $\tilde{\Psi}_e$  is the Fourier transform of  $\tilde{\psi}_e$ , defined through

$$\tilde{\Psi}_e(z, \Omega) = \int_{-\infty}^{\infty} \tilde{\psi}_e(z, t) e^{-j\Omega t} dt. \quad (4-3)$$

Now upon taking the Laplace transform *w.r.t.z*, we obtain from Eq. (4-2),

$$\begin{aligned} s\overline{\Psi}_{e1}(s, \Omega) - \tilde{\Psi}_{e1}(0, \Omega) + j\frac{\Omega}{v}\Psi_{e1} - j\tilde{\delta}\overline{\Psi}_e &= 0, \\ s\overline{\Psi}_{e4}^*(s, \Omega) - \tilde{\Psi}_{e4}^*(0, \Omega) + j\frac{\Omega}{v}\overline{\Psi}_{e4}^* - j\tilde{\delta}\overline{\Psi}_{e1} &= 0 \end{aligned} \quad (4-4)$$

where  $s$  is the Laplace transform variable corresponding to  $z$  and  $\overline{\Psi}_e$  is the Laplace transform of  $\tilde{\Psi}_e$ . These two equations can be simultaneously solved to determine  $\overline{\Psi}_{e4}^*(s, \Omega)$ :

$$\overline{\Psi}_{e4}^*(s, \Omega) = \frac{s + j\Omega/v}{s^2 + \beta^2} \tilde{\Psi}_{e4}^*(0, \Omega) - \frac{j\tilde{\delta}}{s^2 + \beta^2} \tilde{\Psi}_{e1}(0, \Omega) \quad (4-5)$$

where

$$\beta^2 = \left(\frac{\Omega}{v}\right)^2 + \tilde{\delta}^2. \quad (4-6)$$

The inverse Laplace transform is

$$\begin{aligned} \tilde{\Psi}_{e4}^*(z, \Omega) &= \left\{ \cos\beta z + j\frac{\Omega/v}{\beta} \sin\beta z \right\} \tilde{\Psi}_{e4}^*(0, \Omega) \\ &\quad - j\frac{\tilde{\delta}}{\beta} \sin\beta z \tilde{\Psi}_{e1}(0, \Omega). \end{aligned} \quad (4-7)$$

Upon using the boundary conditions, viz.,

$$\begin{aligned} \tilde{\Psi}_{e1}(0, \Omega) &= F(\Omega) = f_t\{f(t)\} \\ \tilde{\Psi}_{e4}^*(L, \Omega) &= 0 \end{aligned} \quad (4-8)$$

we can solve for  $\tilde{\Psi}_{e4}^*(0, \Omega)$  in terms of  $F(\Omega)$  using Eq. (4-7). Thereafter, we can write

$$\tilde{\Psi}_{e4}^*(z, \Omega) = H(z, \Omega)F(\Omega) \quad (4-9)$$

where  $H(z, \Omega)$  represents a transfer function for phase conjugation and is given by

$$H(z, \Omega) = j\tilde{\delta} \frac{\sin\beta(L - z)}{\beta\cos\beta L + j(\Omega/v)\sin\beta L}. \quad (4-10)$$

Of interest to us is the phase conjugate at  $z = 0$ . Accordingly, we are interested in  $H(0, \Omega)$ , which reads

$$H(0, \Omega) = -j\tilde{\delta} \frac{\sin\beta/L}{\beta\cos\beta L + j(\Omega/v)\sin\beta L} \quad (4-11)$$

where  $\beta$  is defined in Eq. (4-6).  $H(0, \Omega)$  will henceforth be simply called the *phase conjugate transfer function*. The inverse transform  $h(0, t)$  will give the *impulse response* of the system, or, equivalently, the nature of the conjugate corresponding to a  $\delta$ -function probe excitation. The impulse response is difficult to calculate. We quote here the final result taken from Fisher et al. (1981) and Rigord et al. (1986). This reads

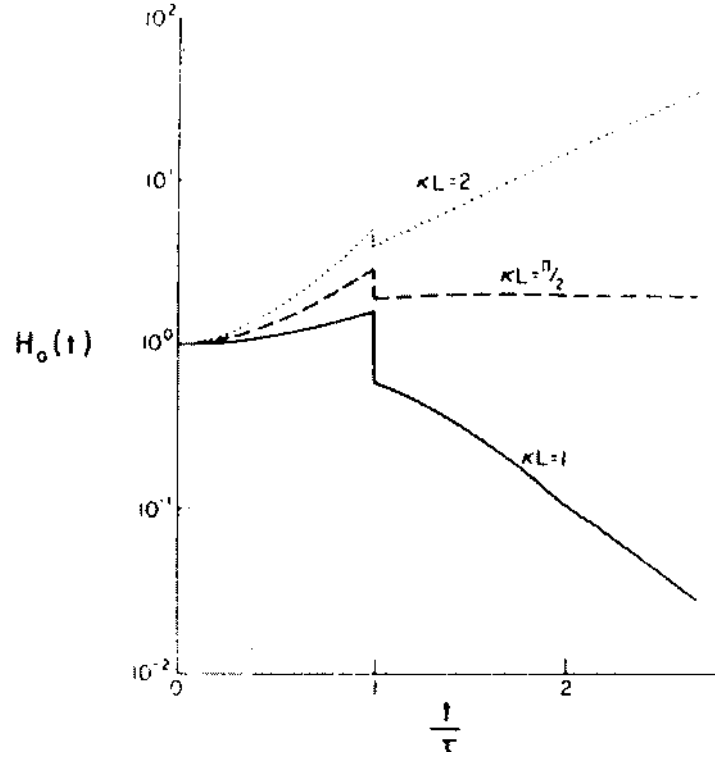
$$\begin{aligned} h(0, t) = -j\frac{\tilde{\delta}v}{z} \Big\{ [I_0(\tilde{\delta}vt) - I_2(\tilde{\delta}vt)]u(t) \\ - \sum_{n=1}^{\infty} [(r_n)^{n-1} I_{2n-2}(t_n) - 2(r_n)^n I_{2n}(t_n) + (r_n)^{n+1} I_{2n+2}(t_n)] \\ \times u(t_n - n\tau) \Big\} \end{aligned} \quad (4-12)$$

where

$$\begin{aligned} t_n &= \tilde{\delta}v(t^2 - n^2\tau^2)^{1/2}; \\ I_n &= \text{modified Bessel function of order } n \\ u(t) &= \text{unit step function} \\ \tau &= \text{round-trip transit time} = 2L/v. \end{aligned} \quad (4-13)$$

Fig. 9 shows this expression evaluated for three cases:  $\tilde{\delta}L = 1$ ,  $\pi/2$ , and 2.

The  $\delta$ -function response exhibits many interesting features. The discussion below closely follows Sudyam and Fisher (1984). Obviously, there is no response until the  $\delta$ -function pulse strikes the entrance face of the conjugator. At that time, the conjugate signal abruptly rises, and thereafter gradually rises until the round-trip time of the conjugator. At the time  $2L/v$ , the conjugate signal abruptly drops to a lower value, and thereafter rapidly approaches exponential decay for  $\tilde{\delta}L < \pi/2$ , converges to a steady state value at twice its initial value for  $\tilde{\delta}L = \pi/2$ , or increases without limit for  $\tilde{\delta}L > \pi/2$ . The explanation for the gentle rise during the first round-trip transit time is simple: As the  $\delta$ -function pulse passes through the conjugator, it generates a backward conjugate wave. Because of the narrow-band reflectivity of the conjugator, the backward-wave spectrum is peaked at the pump-wave frequency of the conjugator (in contrast to the flat spectrum associated with the forward-traveling  $\delta$  function). As the backward wave travels through the



**Figure 9** Plot of backward-wave conjugate impulse response. The solid curve illustrates  $h(0,t)$  for stable conjugation, with  $\tilde{\delta}L=1$ ; the dashed curve, the critical case of oscillation threshold, with  $\tilde{\delta}L=\pi/2$ ; and the dotted curve, an unstable case, with  $\tilde{\delta}L=2$ . In all cases, the normalized curve exhibits a discontinuous drop of value 1 at the end of the first round-trip transit time. [From Sudyam and Fisher (1983)].

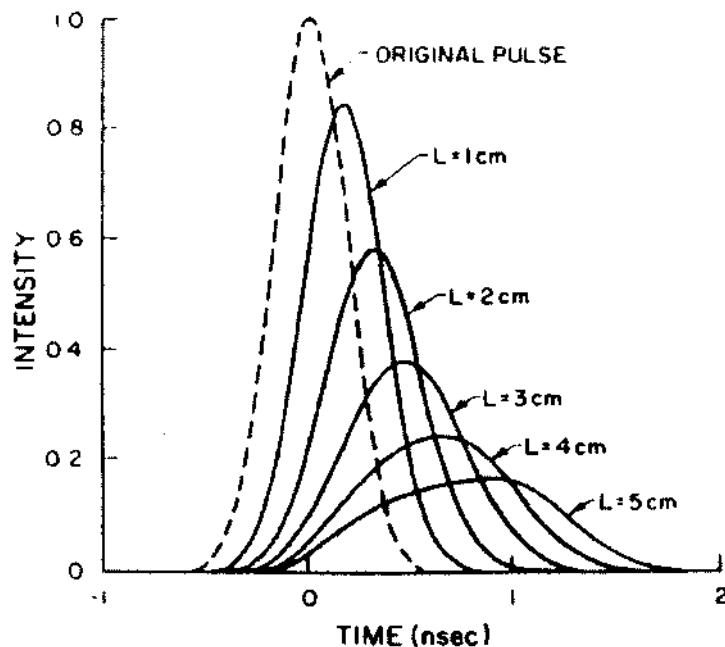
conjugator on its way out, it, too, becomes conjugated into a forward-going wave. This conjugation and reconjugation continuously couples forward- and backward-going waves, resulting in a smooth tail on the forward-going  $\delta$  function. It is the growth of this smooth tail that accounts for the increase in conjugate intensity as the  $\delta$  function pulse traverses the device.

As the  $\delta$ -function pulse leaves the conjugator, only the temporally smooth backward- and forward-going waves remain inside: The  $\delta$  function no longer generates a “new” backward wave. This accounts for the sudden discontinuous decrease in the conjugate signal at the time when an observer at the entrance face could first learn that the  $\delta$  function has left the sample. For  $\tilde{\delta}L < \pi/2$ , the coupling is inadequate for the remaining waves (both forward and backward) rapidly approaching an exponential decay. For  $\tilde{\delta}L \geq \pi/2$ , the

regeneration continues even after the  $\delta$  function leaves the conjugator, giving a steady state ( $\tilde{\delta}L = \pi/2$ ) or exponentially growing ( $\tilde{\delta}L > \pi/2$ ) output.

For  $\tilde{\delta}L \ll 1$ , the  $\delta$ -function response is merely a flat-topped function that turns on at  $t = 0$  and off at the round-trip transit time. This can also be easily understood in the partial-wave picture. The coupling is so weak that the primary backward-going wave is not reconjugated, and therefore no tail grows on the forward-going  $\delta$  function. Thus the radiated intensity remains constant until the round-trip transit time, and thereafter the output is zero because the reconjugation of the partial waves can be neglected.

Because a temporal  $\delta$  function is converted into a conjugate pulse with a duration longer than the round-trip transit time of the conjugator, it is interesting to examine the influence of this temporal spreading upon physical pulses. Fig. 10 shows the conjugate pulse when a 0.5-nsec (full



**Figure 10** Temporal spreading of the conjugate pulse. This calculation was performed by keeping  $\tilde{\delta}L$  equal to  $\pi/4$  (so that the conjugate reflectivity on resonance is unity), but with  $L$  (and, correspondingly  $\tilde{\delta}$ ) variable. This procedure varies the effective bandwidth of the conjugator. The original pulse 0.5 nsec of width is shown by the dashed line. The conjugator material has a refractive index of 4.0, as is the case for germanium. [From Fisher (1983)].

width at  $1/e$  of maximum) Gaussian pulse is conjugated by a conjugator with a linear index of 4.0 (as would be the case for germanium). The parameter  $\tilde{\delta}L$  has been set equal to  $\pi/4$ , but  $L$  and, correspondingly,  $\tilde{\delta}$ , are variable. Thus the conjugate reflectivity on resonance is unity, but the effective bandwidth of the conjugator is varied. In the figure, the input pulse (with carrier frequency chosen equal to the pump-wave frequency) is shown for comparison. The peak intensity decreases and the temporal duration increases as the physical thickness of the conjugator is varied from 1 to 5 cm. These results can be explained by noting that the induced gratings consist of more and more lines as the conjugator becomes longer. Hence the bandwidth of the conjugator is reduced to such an extent that it cannot efficiently conjugate all the frequency components in the input pulse. Note that the delay is approximately half the conjugator round-trip transit time.

## 5 DISCUSSION

It is clear that phase conjugate optics is an important area in the field of nonlinear optics. Typical applications include wave-front correction for waves or images traveling through a distorting medium [Yariv (1977)], compensation of pulse distortion in optical fibers [Yariv et al. (1979)], convolution and correlation of images [White and Yariv (1980)], real-time adaptive optics, laser resonator, optical communication and image transmission etc. [Zeldovich et al. (1985), Rockwell (1988), Shen (1984), Agarwal et al. (1983)]. Self-pumped phase conjugation, phase conjugation, and self-phase conjugation using photorefractive materials will be discussed more in Chap. 9.

## 6 PROBLEMS

1. Consider a medium whose permittivity depends on the field intensity that is

$$\varepsilon(\mathbf{r}) = \varepsilon_0(\mathbf{r}) + \varepsilon_2(\mathbf{r}) |E|^2.$$

Show that the wave equation

$$\nabla^2 E - \frac{\mu}{c^2} \left[ \varepsilon_0(\mathbf{r}) + \varepsilon_2(\mathbf{r}) |E|^2 \right] \frac{\partial^2 E}{\partial t^2} = 0$$

is satisfied by

$$E = [\psi(\mathbf{r})e^{-jkz} + af(\mathbf{r})e^{+jkz}]e^{j\omega t} + c.c.$$

provided

$$f(\mathbf{r}) = \psi^*(\mathbf{r}) \text{ and } a = 1,$$

and for real  $\varepsilon$ .

2. Derive the coupled-mode equations during phase conjugation in the case where the frequency of the probe beam  $\omega_1$  is related to that of the pumps ( $\omega_2 = \omega_3 = \omega$ ) by

$$\omega_1 = \omega - \delta\omega.$$

- (a) Show that the conjugate wave has a frequency  $\omega_4 = \omega + \delta\omega$ .
  - (b) Solve the couple-mode equations for the reflection coefficient  $|A_4(0)/A_1(0)|^2$ , and plot as a function of the frequency offset  $\delta$ .
3. In the presence of losses, the coupled equation describing phase conjugation are

$$\begin{aligned} \frac{dA_1^*}{dz} &= -j\kappa e^{-\alpha L} A_4 - \alpha A_1^* \\ \frac{dA_4}{dz} &= -j\kappa e^{-\alpha L} A_1^* - \alpha A_4 \end{aligned}$$

where  $\kappa$  is proportional to  $\chi^{(3)}$  and  $\alpha$  is the loss coefficient. Solve for the conjugate wave and discuss the nature of the solution.

4. Show that the ABCD matrix [Poon and Banerje (2001)] for a phase conjugate “mirror” can be written as

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and that the q-transformation of a Gaussian beam after reflection from a phase conjugate mirror is given by

$$q_{\text{after}} = -q_{\text{before}}^*.$$

What is the advantage of having a phase conjugate mirror instead of a conventional mirror in a cavity?

5. Show how a “thin” sample of a  $\chi^{(3)}$  material can be used to generate the phase conjugate of an object. Hint: Consider a reference beam such as a plane wave and a point object to be incident on the material and show how a real image of the object can be formed behind the nonlinear material.
6. In the case of unequal pump intensities, solve the set of Eqs. (3–7b) and (3–7c) and derive the spatial variation of the pumps. Can an equivalent propagation constant  $\bar{k}_0$  be defined in this case?

7. Investigate forward phase conjugation in a thin slice of a  $\chi^{(3)}$  material when nominally copropagating object beam and a reference wave are incident on the material. Show that in this case, phase conjugation of the transverse phase is seen in the first non-Bragg diffracted order.

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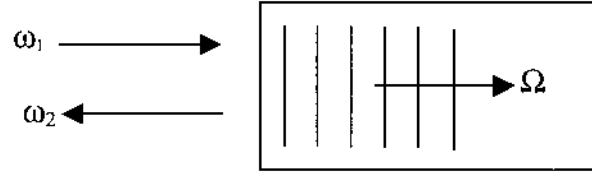
## Stimulated Brillouin and Raman Scattering

In previous chapters, we have examined both second- and third-order nonlinearities and some of the effects stemming from these nonlinearities. In all third-order nonlinear processes, the mechanism for nonlinearity was assumed to originate from the instantaneous nonlinear response of the polarization to the incident optical field. In this chapter, we will examine further causes of the origin of the effective nonlinear optical response in some materials. All of these factors are associated with the scattering of light from the materials because of the induced material response.

### 1 STIMULATED BRILLOUIN SCATTERING

In a loose sense, *stimulated Brillouin scattering* (SBS) can be referred to as nonlinear collinear *acousto-optics*. In a traditional acousto-optic scheme, a sound wave causes a strain wave in the material, thereby changing its refractive index. If the change in the refractive index is a traveling periodic wave, the laws of momentum and energy conservation dictate the direction and frequency of the diffracted or scattered light. In some cases, the incident light at *Bragg angle* is frequency shifted and efficiently diffracted into the first order. In this configuration, the interaction is noncollinear. Collinear interactions are also possible in some cases, where the incident light is introduced along the direction of propagation of the acoustic wave, and the back-scattered wave is frequency shifted.

Consider the schematic shown in Fig. 1. An incident coherent wave of frequency  $\omega_1$  scatters from a periodic acoustic wave of frequency  $\Omega$  to give rise to a scattered light of frequency  $\omega_2 = \omega_1 - \Omega$  (Boyd, 1992). This scattered light is also referred to as the *Stokes wave*. The resulting scattered optical wave as a result of what is called *stimulated Brillouin scattering* (SBS) is Doppler shifted



**Figure 1** Stimulated Brillouin scattering (SBS).

in frequency. There is some interesting physics concealed in the relation mentioned above. It can be shown that the above relation corresponds to stimulated phonon emission. Indeed, amplification of a sound wave has been demonstrated in this case (Korpel et al., 1964).

A physical process by which the interference of the incident laser wave and the Stokes wave drives the acoustic wave is called *electrostriction*, which is the tendency of materials to become more dense, and hence, more optically dense, in regions of high optical intensity. The origin of this effect can be explained via the tendency of a dielectric to be attracted in the direction of an increasing electric field. The situation is similar to what happens when, after combing one's hair on a dry day, the comb picks up pieces of paper. There are polarization charges of both signs induced in the dielectric, which are attracted and repelled by the electric field. There is a net attraction because the field nearer to the comb is stronger than the field farther away from it. In fact, *for small objects, the force is proportional to the square of the electric field*. The quadratic nature stems from the fact that the induced polarization is proportional to the electric field and the force is likewise proportional to the field. Furthermore, the force is proportional to the gradient because, as indicated above, there is net attraction only if the field varies from point to point. A detailed analysis of this effect is outside the scope of this book; interested readers can refer to Boyd (1992) and Yariv (1991).

Because of the attraction of particles toward a region of higher electric field, the density, and hence the susceptibility, of the material increases, leading to an increased refractive index. The induced electrostrictive change in the refractive index is given by

$$\Delta n \approx n_0(\Delta\epsilon/2\epsilon_0) = (n_0/2\epsilon_0)(\partial\epsilon/\partial\rho)\Delta\rho. \quad (1-1)$$

Also, by using energy concepts, the change in density  $\Delta\rho$  due to electrostriction can be calculated as (Boyd, 1992)

$$\Delta\rho = \rho_0 C \gamma_e \langle E^2 \rangle \quad (1-2)$$

where  $C$  is proportional to the *compressibility* of the medium. The constant  $\gamma_e = (\partial\epsilon/\partial\rho)\rho_0$  is called the *electrostrictive constant*, where  $\rho_0$  refers to the

unperturbed density of the medium. Using Eq. (1-2) and the definition of  $\gamma_e$ , we can rewrite Eq. (1-1) as

$$\Delta n \approx (n_0/2\epsilon_0)C\gamma_e^2\langle E^2 \rangle. \quad (1-3)$$

Consider, again, the situation depicted in Fig. 1. Here, an incident optical field  $E_1$  of angular frequency and wavenumber  $(\omega_1, \mathbf{k}_1)$  interacts with an acoustic wave of amplitude  $\rho$  and of angular frequency and wavenumber  $(\Omega_B, \mathbf{K}_B)$  to generate a counterpropagating optical field  $E_2$  of angular frequency and wavenumber  $(\omega_2, \mathbf{k}_2)$ . From the conservation of momentum for the photon-phonon downshifted interaction,

$$\hbar \mathbf{k}_1 + \hbar \mathbf{K}_B = \hbar \mathbf{K}_B + \hbar \mathbf{K}_B + \hbar \mathbf{k}_2. \quad (1-4)$$

Assuming that  $\mathbf{k}_2 \approx -\mathbf{k}_1$ , Eq. (1-4) implies that

$$\mathbf{K}_B = 2 \mathbf{k}_1 \quad (1-5)$$

or equivalently

$$\Omega_B = 2\omega_1(V/v) \quad (1-6)$$

where  $V$  and  $v$  denote the velocity of the sound and light waves in the medium, respectively.  $\Omega_B$  is referred to as the *Brillouin frequency*.

In general, interactions between two light waves at frequencies  $\omega_1, \omega_2$  will give rise to a sound wave at frequency  $\Omega = \omega_1 - \omega_2$ . However, the acoustic wave will be efficiently generated if  $\Omega$  is close to the Brillouin frequency  $\Omega_B$ .

In order to study the interaction between the optical and acoustic fields, we need to write down the optical equation along with the material (acoustic) equation. The optical field obeys the equation (Poon and Banerjee, 2001)

$$\partial^2 E / \partial t^2 - v^2 \partial^2 E / \partial z^2 = -2(\Delta n / n_0) \partial^2 E / \partial t^2. \quad (1-7)$$

This equation is readily derived from the conventional wave equation, assuming that the refractive index  $n$  is now expressible as  $n = n_0 + \Delta n$ , where  $\Delta n$  is the induced change in refractive index. The induced refractive index change is proportional to the density and hence to the pressure wave in the medium, satisfying the acoustic wave equation

$$\partial^2 \Delta n / \partial t^2 - V^2 \partial^2 \Delta n / \partial z^2 - \alpha_s \partial n / \partial t = F \partial \Delta^2 \langle E^2 \rangle / \partial z^2 \quad (1-8)$$

where  $\alpha_s$  is a dissipation constant accounting for acoustic losses, and  $F$  is a constant proportional to the compressibility and the electrostrictive constants defined above. In Eq. (1-8), the right hand side (RHS) is the driving term, and as noted earlier, it originates from the fact that the force is proportional to the gradient of the time-averaged square of the optical field. We would like to point out that the last term on the left hand side (LHS) of Eq. (1-8) is

according to the discussion in Yariv (1991). In Boyd (1992), this term is written as proportional to  $\partial^3 \Delta n / \partial z^2 \partial t$ ; however, this does not add complexity to the problem.

We now assume

$$E = \frac{1}{2} [A_1(z) \exp j(\omega_1 t - k_1 z) + A_2(z) \exp j(\omega_2 t - k_2 z) + \text{c.c.}] \quad (1-9)$$

and

$$\Delta n = \frac{1}{2} \delta n \exp j(\Omega t - Kz) + \text{c.c.} \quad (1-10)$$

and first substitute in Eq. (1-8). The term  $\partial^2 \langle E^2 \rangle / \partial z^2$  on the RHS is equal to  $\frac{1}{2} A_1 A_2^* \exp j(\Omega t - Kz) + \text{c.c.}$ , where time averaging has been performed over the period of the optical field, and where we have set  $\Omega = \omega_1 - \omega_2$ ,  $K = k_1 - k_2$ . Hence from Eq. (1-8), we obtain

$$\delta n = j(FK/\alpha_s V) A_1 A_2^* \quad (1-11)$$

where we assumed  $\Omega/K = V$ , the velocity of the acoustic wave. This also implies that the refractive index grating is out of phase with the intensity grating by  $90^\circ$ . The situation is similar to that in photorefractive materials with diffusion dominated nonlinearity, as discussed in Chap. 9, and can lead to energy exchange between the interacting optical waves through the induced refractive index grating. Using Eqs. (1-10) and (1-11), it follows that

$$\Delta n = -\frac{1}{2} j(FK/\alpha_s V) A_1 A_2^* \exp j(\Omega t - Kz) + \text{c.c.} \quad (1-12)$$

We can now substitute Eqs. (1-9) and (1-12) into Eq. (1-7) to derive the spatial evolution equations for the optical wave envelopes  $A_1, A_2$ . After some algebra and upon equating the coefficients of  $\exp j(\omega_1 t - k_1 z)$  and  $\exp j(\omega_2 t - k_2 z)$ , respectively, we obtain the following set of coupled equations:

$$\frac{dA_1}{dz} = -\frac{FK}{2v^2 n_0 k_1 \alpha_s V} A_1 |A_2|^2 \quad (1-13a)$$

$$\frac{dA_2}{dz} = -\frac{FK}{2v^2 n_0 k_2 \alpha_s V} A_2 |A_1|^2. \quad (1-13b)$$

Putting  $k_2 = -k_1$  (considering that the pump and SBS waves are contra-propagating), we obtain

$$\frac{dA_1}{dz} = -\frac{\Gamma_B}{2} A_1 |A_2|^2 \quad (1-14a)$$

$$\frac{dA_2}{dz} = -\frac{\Gamma_B}{2} A_2 |A_1|^2 \quad (1-14b)$$

where

$$\Gamma_B = \frac{FK}{v^2 n_0 k_2 \alpha_s V}. \quad (1-15)$$

The constant  $\Gamma_B$  defined in Eq. (1-15) is referred to as the SBS *gain factor*. Notice that Eqs. (1-14a) and (1-14b) do not have any  $j$  on the RHS, hence we can assume  $A_{1,2}$  to be real without loss of generality. Also, writing  $I_{1,2} = A_{1,2}^2$ , we can recast Eqs. (1-14a) and (1-14b) in the form

$$dI_1/dz = -\Gamma_B I_1 I_2 \quad (1-16a)$$

$$dI_2/dz = -\Gamma_B I_1 I_2. \quad (1-16b)$$

It is clear from Eq. (1-16a, 1-16b) that

$$I_1 - I_2 = \text{const} = C = I_1(0) - I_2(0) \quad (1-17)$$

from conservation of energy, which is true for coupling between optical waves through induced reflection gratings, similar to what happens in photorefractive materials. Using Eq. (1-17), Eq. (1-16b) can be integrated as

$$\int_{I_2(0)}^{I_2(z)} \frac{dI_2}{I_2(I_2 + C)} dI_2 = -\Gamma_B z$$

which yields, after some algebra (Boyd, 1992),

$$I_2(z) = \frac{CI_2(0)}{I_1(0)\exp(CT_B z) - I_2(0)} \quad (1-18)$$

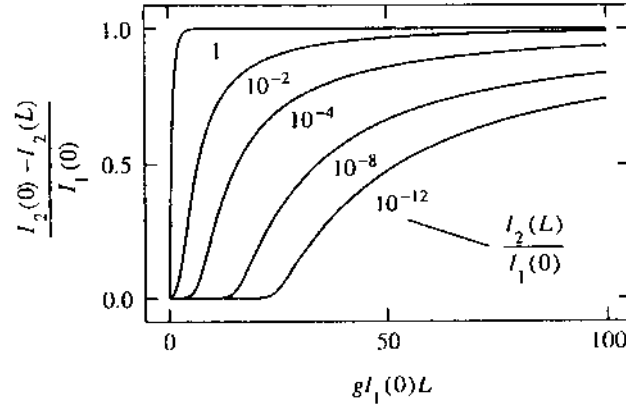
where  $C$  is defined in Eq. (1-17). The expression for  $I_1$  readily follows from Eq. (1-17).

Note, however, that the value of  $I_2$  is really specified at  $z = L$ , rather than at  $z = 0$ . To relate  $I_2(0)$  to  $I_2(L)$ , we need to set  $z = L$  in Eq. (1-18), and solve the resulting transcendental equation. The typical plot of  $I_2(0)$  as a function of  $L$  is shown in Fig. 2. An approximate expression for  $I_2(0)$  can be obtained for small conversion ratio  $I_2(0)/I_1(0)$  as

$$I_2(0) = I_2(L)\exp(I_1(0)\Gamma_B L), \quad (1-19)$$

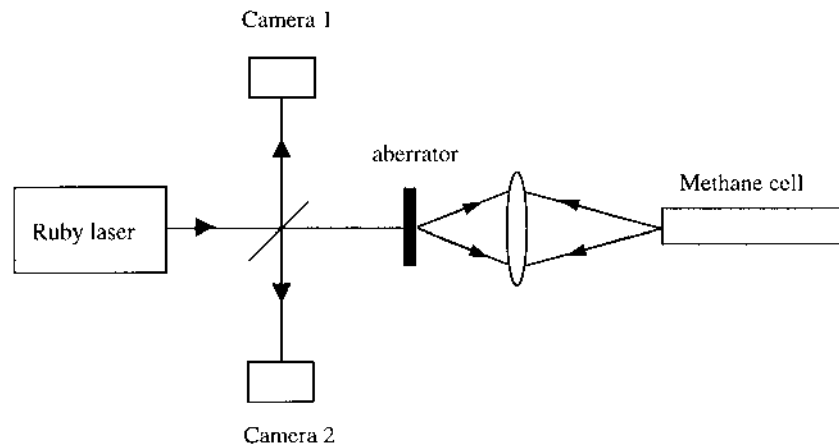
which shows that the initial increase is exponential in nature, as also demonstrated in Fig. 2.

It has been shown that SBS can be used for *self phase conjugation* (Zeldovich et al., 1972). In a typical experiment, laser light passing through an aberrator is introduced into an SBS medium, as shown in Fig. 3. The retroreflected SBS beam retraces through the aberrator and is monitored. The results indicate that the uncorrected aberrated beam can be corrected, as



**Figure 2** Intensity gain characteristics during stimulated Brillouin scattering [Boyd (1992)]. The  $g$  in the figure corresponds to the gain factor  $\Gamma_B$  in the text.

shown in the inset of Fig. 3. In this case, there is no Stokes field externally injected into the interaction region. Hence the value of  $I_2(L)$  is not known a priori. Therefore, the effective value of  $I_2(L)$  is proportional to the local value of  $I_1(L)$  in the medium. Note that the generated SBS wave of frequency  $\omega_2$  is different than that for the incident wave of frequency  $\omega_1$ , by the amount  $\Omega_B = \omega_1 - \omega_2$ , which is the Brillouin frequency. SBS in other gases have been experimentally demonstrated, for details, the reader is referred to Damzen and Hutchinson (1983).

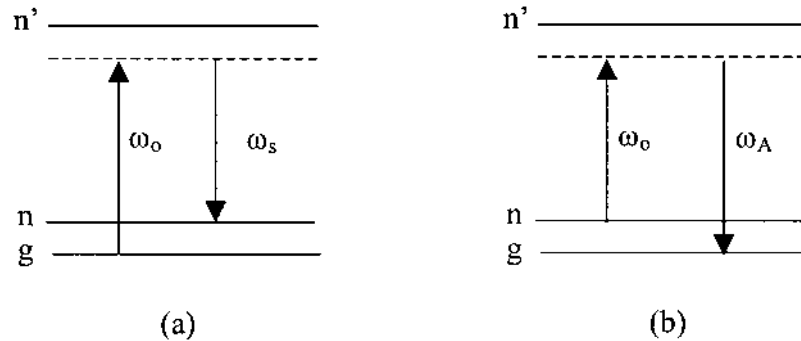


**Figure 3** Schematic of phase conjugation using SBS.

## 2 STIMULATED RAMAN SCATTERING

The *spontaneous Raman effect* was discovered by Raman in 1928. A light beam of frequency  $\omega_0$  illuminating a material sample which forms the Raman scattering medium, scatters light at frequencies  $\omega_S$  and  $\omega_A$ , referred to as the *Stokes* and *anti-Stokes* frequencies, respectively. Raman or Stokes scattering results from the initial excitation of the material from ground state to an excited state by absorbing a photon of frequency  $\omega_0$ , and subsequent radiation of a photon at a lower frequency  $\omega_S = \omega_0 - \omega_V$ . The remaining energy at frequency  $\omega_V$  is transferred to internal material excitation, such as lattice vibration, etc. (see Fig. 4a). If the material is already in an excited state, the incident photon at frequency  $\omega_0$  can lead to a scattered photon at frequency  $\omega_A = \omega_0 + \omega_V$ , as shown in Fig. 4b. Usually, anti-Stokes scattering is weaker than Stokes scattering by a factor  $\exp(-h\omega_V/k_B T)$ , where  $k_B$  is the Boltzmann constant (Boyd, 1992).

In *stimulated Raman scattering* (SRS), both optical frequencies  $\omega_0$  and  $\omega_S$  are incident on the medium, resulting in the amplification of Stokes photons. SRS was first observed by Woodbury and Ng (1962), and its properties investigated in detail by Bloembergen (1967). In practice, anti-Stokes photons may also be generated during the process. The physics of the SRS process can be understood as follows (Hellwarth, 1963). The normal mode of the molecules in the material oscillating at a natural frequency  $\omega_V$  beats with the incident laser frequency, thus generating photons at frequency  $\omega_S = \omega_0 - \omega_V$ . Concurrently, the incident photons and the Stokes photons produce a beat frequency  $\omega_V$ , which drives the normal mode vibration of the molecules. The reinforced molecular vibration, in turn, coherently amplifies the Stokes wave. SRS, leading to Stokes wave amplification (and consequent



**Figure 4** Energy level diagrams showing (a) Stokes scattering and (b) anti-Stokes scattering.

depletion of the incident laser wave at frequency  $\omega_0$ ), can be modeled by using an *effective third-order nonlinear Raman susceptibility*  $\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S)$ . Using the concepts from Chap. 2, it follows that

$$\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S) = \chi_R^{(3)*}(-\omega_0; \omega_S, -\omega_S, \omega_0). \quad (2-1)$$

The coupling of optical waves with complex amplitudes  $A_P$ ,  $A_S$  defined through the expression for the total optical field  $E$  as:

$$E = \frac{1}{2} [A_P \exp(\omega_0 t - k_0 z) + A_S \exp(\omega_S t - k_S z) + \text{c.c.}] \quad (2-2)$$

can be understood through the coupled spatial evolution equations

$$\frac{dA_S}{dz} = -jk_0 \chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S) A_S |A_P|^2 \quad (2-3a)$$

$$\frac{dA_P}{dz} = -jk_0 \chi_R^{(3)}(-\omega_0; \omega_S, -\omega_S, \omega_0) A_S |A_P|^2. \quad (2-3b)$$

It turns out that  $\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S)$  has a real and an imaginary part, when plotted as a function of the Stokes frequency  $\omega_S$ . The frequency dependence of  $\chi_R^{(3)}$  can be obtained by using a heuristic model of a damped harmonic oscillator, as in the case of Chap. 2. It also depends on the internuclear distance, the damping constant, the pump or excitation optical frequency  $\omega_0$ , and the vibrational frequency  $\omega_V$ . Around the frequency  $\omega_S = \omega_0 - \omega_V$ , the real part of  $\chi_R^{(3)}$  is zero, while the imaginary part is a maximum. Writing  $\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S) = j\chi_R^{(3)''}(-\omega_S; \omega_0, -\omega_0, \omega_S)$  for this case, it then follows that  $\chi_R^{(3)}(-\omega_0; \omega_S, -\omega_S, \omega_0) = j\chi_R^{(3)''}(-\omega_0; \omega_S, -\omega_S, \omega_0) = -j\chi_R^{(3)''}(-\omega_S; \omega_0, -\omega_0, \omega_S)$ . Thus, substituting this into Eqs. (2-3a) and (2-3b), we obtain (Sutherland, 1996)

$$\frac{dA_S}{dz} = \frac{\Gamma_R}{2} A_S A_P^2, \quad (2-4a)$$

$$\frac{dA_P}{dz} = -\frac{\Gamma_R}{2} A_P A_S^2, \quad (2-4b)$$

where

$$\Gamma_R = 2k_0 \chi_R^{(3)''}(-\omega_S; \omega_0, -\omega_0, \omega_S) \quad (2-5)$$

is the *SRS gain coefficient*, and where we assumed  $A_P$ ,  $A_S$  to be real, without loss of generality. Defining  $I_{P,S} = A_{P,S}^2$ , we can recast Eqs. (2-4 a, 2-4b) in the form

$$\frac{dI_S}{dz} = \Gamma_R I_S I_P, \quad (2-6a)$$

$$\frac{dI_P}{dz} = -\Gamma_R I_S I_P. \quad (2-6b)$$

Note that Eqs. (2-6a) and (2-6b) are similar to Eqs. (1-16a) and (1-16b), which defined two-wave coupling between contrapropagating optical waves. In this case, Eqs. (2-6a) and (2-6b) model two-wave coupling between copropagating waves, much like two-wave coupling due to transmission gratings in a photorefractive material, which will be discussed further in Chap. 9. It follows from conservation of energy that

$$I_S + I_P = I_S(0) + I_P(0) = \text{const} = C. \quad (2-7)$$

The solutions of Eqs. (2-6a) and (2-6b) are of the form

$$I_S = I_S(0) \frac{1+m}{1+m \exp(-\Gamma_R z/C)}, \quad (2-8a)$$

$$I_P = I_P(0) \frac{1+m^{-1}}{1+m^{-1} \exp(\Gamma_R z/C)}, \quad (2-8b)$$

where  $m = I_P(0)/I_S(0)$ . The solutions are similar to that observed in photorefractive two-wave mixing in the presence of induced transmission gratings, and are further discussed in Chap. 9.

As pointed out earlier, both Stokes and anti-Stokes waves may be generated in the process. The spatial evolution of the Stokes and anti-Stokes waves can be derived in a similar fashion, for the general case of depleted pump, as well as under the approximation of undepleted pump. The relative phase mismatch between the participating waves is an important parameter that determines the amplification of the Stokes and anti-Stokes waves. It turns out that efficient Stokes wave generation, along with attenuation of the anti-Stokes wave, can be achieved by propagating the Stokes wave at an angle with respect to the pump wave. Details of this are outside the scope of this work and can be found in Boyd (1992).

### 3 PROBLEMS

1. With absorption included in the evolution equations, Eqs. (1-16a) and (1-16b) are modified to

$$dI_1/dz = -\Gamma_B I_1 I_2 - \alpha I_1, \quad dI_2/dz = -\Gamma_B I_1 I_2 + \alpha I_2$$

where  $\alpha$  is the absorption coefficient. Show that now,  $I_1 - I_2 = [I_1(0) - I_2(0)] \exp(-\alpha z)$ . For the case of the undepleted pump, find the expression for the SBS intensity. What is the maximum allowed value of  $\alpha$  to observe net amplification of the SBS signal?

2. Find an estimate of the SBS reflectivity  $R = I_2(0)/I_1(0)$ , assuming that  $I_2(L)$  is small. Plot the reflectivity as a function of the SBS gain factor.
3. Using momentum diagrams similar to those used to study phase conjugation of pulses, investigate the pulse duration of the Stokes radiation excited by SBS when the pump is pulsed. Moreover, investigate the nature of SBS when the pump has a transverse amplitude/phase profile.
4. In the general case, when  $\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S)$  is complex, find the reduced set of coupled equations in terms of the real and imaginary parts of  $\chi_R^{(3)}(-\omega_S; \omega_0, -\omega_0, \omega_S)$ .

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# 8

## Solitons in Optical Fibers

In earlier chapters, we have discussed various examples of applications of the induced nonlinear polarization of light due to several different mechanisms. In particular we have discussed some of the effects of the cubic nonlinearity, especially in the context of self-phase modulation (SPM), spatial solitons, optical bistability, and phase conjugation. In this chapter, we will discuss one of the applications of cubic nonlinearity in the area of pulse propagation in optical fibers. We will introduce readers to the concept of *temporal* solitons as opposed to *spatial* solitons, and study the propagation of arbitrary pulses through nonlinear optical fibers.

Fiber-optic communication systems typically operate in the visible or shortwave infrared region at frequencies on the order of several hundred terahertz ( $\sim 300$  THz). Such communication systems rely upon optical fibers for data transmission. Semiconductor lasers or light-emitting diodes are commonly employed optical transmission sources because these devices are readily compatible with the optical fibers used as the communication channel.

One common measure of the information-carrying capacity of the fiber-optic communication system is the product of the bit rate and the fiber unit length, or the *bit rate–distance product*. Since the invention of the laser in 1960 and the subsequent demonstration of early optical fibers as communication media, the bit rate–distance product has increased by many orders of magnitude (Agrawal, 1995). The first real breakthrough in optical communications occurred in 1977 with the development of the InGaAsP semiconductor laser and optical receivers operating at  $1.3\text{ }\mu\text{m}$ , where fiber loss is typically less than 1 dB/km and fiber dispersion is a minimum. Bit rate–distance products of these first-generation lightwave communication systems were generally limited to less than 1 Gbit/sec km (Kapron et al., 1970). Optical solitons in fibers were first observed by Mollenauer et al (1980). Roman gain in the fiber was used for long haul soliton communication systems (Mollenauer and Smith, 1988). Technology advancements over the last 20 years have led to

fourth-generation communication systems, referred to as coherent optical communication systems, and finally to fifth-generation systems, or soliton communication systems (Ippen et al., 1974; Green, 1993). Bit rate–distance products of  $10^5$  Gbits/sec km are feasible using dispersion managed systems with soliton pulses and erbium-doped fiber amplifiers (Stolen and Lin, 1978).

This chapter examines optical fiber characteristics, including dispersion and nonlinearity, and formulates several pulse propagation equations, including the *nonlinear Schrödinger (NLS) equation*, and *self-steepening* equation. *Self-focusing*, an effect observed for beams in a nonlinear medium, and which also involves analysis using the NLS equation, has been discussed in Chap. 4. Section 1 provides a physical description of important optical fiber parameters, including the mathematical background relating to their physical description. Section 2 covers the two important fiber parameters of dispersion and Kerr nonlinearity in detail, parameters that must be understood in order to grasp the concept of optical solitons. *Optical solitons* and the NLS equation are the subject of Sec. 3, and Sec. 4 describes the physical processes that govern self-steepening and self-focusing.

## 1 BACKGROUND ON LINEAR OPTICAL FIBERS

An *optical fiber* generally consists of a center core made of silica glass embedded inside an outer layer of lower refractive index glass known as the cladding. Doping elements, which increase the refractive index of pure silica (such as  $\text{GeO}_2$  and  $\text{P}_2\text{O}_5$ ), are used in the fiber core while elements such as boron and fluorine, which lower the index, are used for the fiber cladding. Optical fibers may be manufactured either as *step-index* fibers, in which there is a sudden change in the refractive index between the core and cladding, or as *graded-index* fibers, for which the fiber index gradually decreases from the center of the core to the cladding interface (Agrawal, 1995).

We will limit ourselves to step-index fibers, which can operate in such a fashion as to carry only one fiber mode, and thus are referred to as single-mode fibers. Fiber modes can be understood by solving *Maxwell's equations*, starting from the time domain, to obtain the basic wave propagation equation in the frequency domain. Maxwell's equations have been enunciated in Chap. 1, in Eqs. (1-1)–(1-4). The *constitutive relations* will be taken as Eq. (1-13a) of Chap. 1, which in conjunction with Eq. (1-15) of Chap. 1 and Table 1 of Chap. 2, is

$$\mathbf{D}(\mathbf{r}, t) = \tilde{\epsilon}(t) * \mathbf{E}(\mathbf{r}, t), \quad \tilde{\epsilon}(t) \xleftrightarrow[\mathfrak{F}^{-1}]{\mathfrak{F}} \epsilon(\omega). \quad (1-1)$$

Equation (1-1) follows the generalized constitutive relation derived from the linear polarization in the time domain, as described in Chap. 2. Similar to

the treatment in Chap. 1, the wave equation in the time domain for a source-free region can be derived as:

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) = \mu \frac{\partial^2}{\partial t^2} [\tilde{e}(t) * \mathbf{E}(\mathbf{r}, t)]. \quad (1-2)$$

We can express Eq. (1-2) in the frequency domain using the relation  $\mu\epsilon(\omega)\omega^2 = n^2(\omega)k_0^2$ . The wave equation becomes

$$\nabla^2 \tilde{\mathbf{E}}(\mathbf{r}, \omega) + n^2(\omega)k_0^2 \tilde{\mathbf{E}}(\mathbf{r}, \omega) = 0, \quad (1-3)$$

where  $\tilde{\mathbf{E}}(\mathbf{r}, \omega)$  is the Fourier transform of  $\mathbf{E}(\mathbf{r}, t)$ . Also,  $n$  is the refractive index in either the core ( $n_1$ ) or cladding ( $n_2$ ) and  $k_0$  is the free space wave number. Expanding Eq. (1-3) in terms of  $(r, \theta, z)$  (conforming to a radially symmetric optical fiber) yields

$$\frac{\partial^2 \tilde{\mathbf{E}}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{\mathbf{E}}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial \theta^2} + \frac{\partial^2 \tilde{\mathbf{E}}}{\partial z^2} + n^2 k_0^2 \tilde{\mathbf{E}} = 0. \quad (1-4)$$

To determine the fiber modes, we write each Cartesian component  $E_{x,y,z}(\mathbf{r}, t)$  of  $\mathbf{E}(\mathbf{r}, t)$  as  $E_{x,y,z}(\mathbf{r}, t) \approx \text{Re}[\psi(r, \theta) \exp j(\omega_0 t - kz)]$ , and substitute in Eq. (1-4) to get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + [n^2(r, \omega_0)k_0^2 - k^2]\psi = 0. \quad (1-5)$$

The solution of Eq. (1-5), using separation of variables, can be expressed in the form

$$\psi(r, \theta) = \begin{cases} \frac{C}{J_1(U)} J_1(Ur/a) \begin{bmatrix} \cos l\theta \\ \sin l\theta \end{bmatrix}, & r < a \\ \frac{C}{K_1(W)} K_1(Wr/a) \begin{bmatrix} \cos l\theta \\ \sin l\theta \end{bmatrix}, & r > a \end{cases}, \quad (1-6)$$

where  $a$  is the fiber core radius,  $l$  is an integer,  $C$  is an arbitrary constant,  $J$  and  $K$  are the Bessel and modified Bessel functions, respectively, and where continuity of the field at the core-cladding interface  $r = a$  has been assumed. The parameters  $U$  and  $W$  are defined as:

$$U = a(k_0^2 n_1^2 - k^2)^{1/2}, \quad W = a(k^2 - k_0^2 n_2^2)^{1/2}. \quad (1-7)$$

Furthermore, continuity of  $\partial\psi/\partial r$  at  $r = a$  leads to the constraint

$$\frac{U J_1'(U)}{J_1(U)} = \frac{W K_1'(W)}{K_1(W)}. \quad (1-8)$$

The  $V$  number of a fiber is defined as:

$$V = (U^2 + W^2)^{1/2} = k_0 a \sqrt{n_1^2 - n_2^2}. \quad (1-9)$$

It turns out that in a step-index fiber with  $V < 2.405$ , only one guided mode is supported. This is commonly referred to as the *single-mode condition*. Details of single-mode propagation in fibers and other modes can be found in Ghatak and Thyagarajan (1989).

## 2 FIBER DISPERSION AND NONLINEARITY

*Fiber-optic solitons* are the result of the interaction between *fiber dispersion* and *nonlinearity*. Therefore, an understanding of soliton physics begins with the analysis of these two important fiber parameters. Fiber dispersion can best be understood by realizing that the mode propagation constant  $k$  is not a constant, but in fact is a complicated, frequency-dependent parameter. A Taylor series expansion of  $k(\omega)$  about the carrier frequency  $\omega_0$  results in (Agrawal, 1995)

$$k(\omega) = k^{(0)} + k'(\omega - \omega_0) + \frac{1}{2}k''(\omega - \omega_0)^2 + \frac{1}{6}k'''(\omega - \omega_0)^3 + \cdots, \quad (2-1)$$

where the dispersion parameters  $k_i$  are defined as

$$\begin{aligned} k^{(0)} &= \text{mode propagation constant,} \\ k' &= \frac{dk}{d\omega} = \text{group velocity parameter,} \\ k'' &= \frac{d^2k}{d\omega^2} = \text{group velocity dispersion parameter,} \\ k''' &= \frac{d^3k}{d\omega^3} = \text{group velocity dispersion slope parameter.} \end{aligned}$$

Eq. (2-1) is similar to the Taylor series expansion for  $w(k)$  in Eq. (5-12) of Chap. 1.

An optical pulse propagating through a single-mode fiber will have a finite spectral width  $\Delta\omega$ , referred to as an optical (spectral) “bundle” or “group.” Because the mode index  $\bar{n}$  is frequency dependent, it is convenient to define a group index by

$$\bar{n}_g = \bar{n} + \omega \frac{d\bar{n}}{d\omega}. \quad (2-2)$$

The pulse envelope thus travels at a group velocity  $v_g = 1/k' (= u_o$  in Eq. (5-12) of Chap. 1). The group velocity  $v_g$  is frequency dependent, leading to different spectral components traveling at slightly different speeds along the fiber, causing the pulse to disperse during propagation. This phenomenon is known as *group velocity dispersion* (GVD) and is quantified by the parameter  $k''$ . *Dispersion* in optical fibers results from the frequency dependence of the index of refraction  $\bar{n}_g$ . *Dispersion* may be classified as *normal-dispersion* (N-D) or

*anomalous-dispersion* (A-D) depending on whether the GVD parameter  $k''$  is positive or negative (see Fig. 1). For  $k'' > 0$ , the fiber operates in the N-D regime, while for  $k'' < 0$  the fiber operates in the A-D regime.

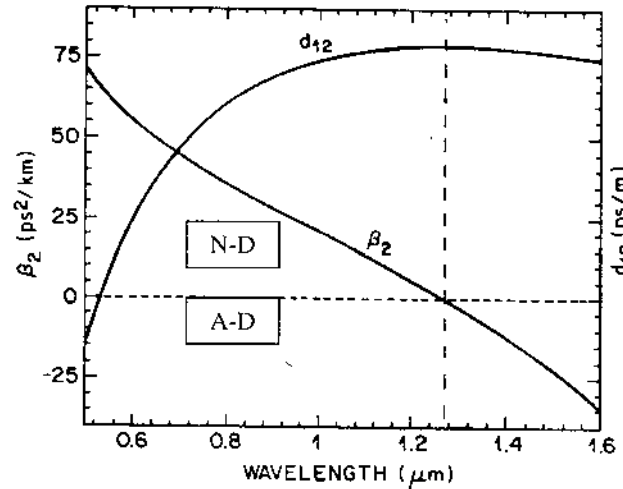
The primary physical difference in the two regions is that under N-D the higher-frequency components travel *slower* than the lower-frequency components, whereas in the A-D regime the effect is reversed, and higher-frequency components travel *faster* than lower-frequency components. The frequency at which the GVD parameter  $k'' = 0$  is called the *zero-dispersion wavelength*.

To derive the propagation of a pulsed optical field in a single-mode fiber taking into account fiber dispersion, we can heuristically express a component of  $\tilde{E}(\mathbf{r}, \omega)$  in the form  $\tilde{a}(z, \omega)\psi(r, \theta)$  where  $\tilde{a}(z, \omega)$  is the longitudinal amplitude with the  $z$  dependence of the form  $\exp -jk(\omega)z$ , where  $k(\omega)$  is given by Eq. (2-1). This alternatively implies

$$\partial \tilde{a} / \partial z = -jk(\omega) \tilde{a}(z, \omega). \quad (2-3)$$

If the spectrum of  $\tilde{a}(z, \omega)$  is shifted to around  $\omega_0$ , the corresponding spatial evolution of the new spectrum  $\tilde{A}(z, \Omega)$ , where  $\Omega = \omega - \omega_0$  obeys the equation

$$\partial \tilde{A}(z, \Omega) / \partial z = -j[k' \Omega + (1/2)k'' \Omega^2 + (1/6)k''' \Omega^3] \tilde{A}(z, \Omega). \quad (2-4)$$



**Figure 1** Variation of GVD parameter with operating frequency [Agrawal (1995)]. The  $\beta_2$  has the same connotation as  $k''$ .

Upon inverse Fourier transforming Eq. (2-4), we get

$$\left( \frac{\partial}{\partial z} + \frac{1}{v_g} \frac{\partial}{\partial T} \right) A(z, T) = j \frac{1}{2} k'' \frac{\partial^2 A(z, T)}{\partial T^2}, \quad (2-5)$$

where  $T$  is the Fourier transform variable corresponding to  $\Omega$ ,  $A(z, T)$  is the inverse transform of  $\tilde{A}(z, \Omega)$ , and where we have left out the effect of  $k'''$  for now for simplicity. Equation (2-5) describes the propagation of pulses through dispersive fibers, and is the linear part of the nonlinear Schrödinger equation.

In Eq. (2-1), the propagation constant  $k$  for a fiber was expressed in terms of a Taylor series depicting its dependence on frequency. Now, if the fiber parameters change because of fiber inhomogenities, it can be shown that the propagation constant can be expressed as (Haus, 1984):

$$k(\omega) = k^{(0)} + k'(\omega - \omega_0) + \frac{1}{2} k''(\omega - \omega_0)^2 + \frac{1}{6} k'''(\omega - \omega_0)^3 + \frac{\omega_0^2 \mu_0 \epsilon_0}{k^{(0)}} \frac{\int dS n \Delta n \psi^2}{\int dS \psi^2} \quad (2-6)$$

where  $\psi$  denotes the transverse profile of a mode of the optical wave in the fiber, described in detail earlier. In addition,  $dS$  denotes the incremental area of the cross section of the fiber. Neglecting  $k'''$  for the sake of simplicity, the corresponding PDE for  $A(z, T)$  becomes:

$$j \frac{\partial A(z, T)}{\partial z} + \frac{1}{2} k'' \frac{\partial^2 A(z, T)}{\partial T^2} - \frac{\omega_0^2 \mu_0 \epsilon_0}{k^{(0)}} \frac{\int dS n \Delta n \psi^2}{\int dS \psi^2} A(z, T) = 0. \quad (2-7)$$

In writing Eq. (2-7), we have switched to a moving frame of reference defined through the transformation  $T' = T - z/v_g$ ;  $z' = z$ , and dropped the primes for convenience.

The effects of fiber nonlinearities are due to intensity dependence of the refractive index for intense light energies. The third-order susceptibility  $\chi^{(3)}$  is the lowest order nonlinear effect present in optical fibers and is the physical component responsible for nonlinear refraction. Thus, the refractive index change in optical fibers may be expressed as

$$\Delta n = n_2 |E|^2, \quad (2-8)$$

where  $n_2$  is the *nonlinear refractive index coefficient*, and  $|E|^2$  is the proportional to the intensity of the light field inside the fiber. The intensity dependence of the refractive index leads to a phenomenon known as *self-phase modulation* (SPM), which manifests itself through a phase shift in the optical pulse as it travels along the fiber. This phase shift is

$$\phi_{nl} = k_0 L n_2 |E|^2. \quad (2-9)$$

Thus, an optical pulse may experience *spectral broadening* as it travels along the fiber due to SPM. However, the pulse shape will remain unchanged in the presence of fiber nonlinearity alone.

If now we incorporate Eq. (2-8) into Eq. (2-7), and note that  $|E|^2 \approx |\mathfrak{F}^{-1}[\tilde{a}(z, \omega)\psi(r, \theta)|^2] = |\mathfrak{F}^{-1}[\tilde{A}(z, \Omega)\psi(r, \theta)]|^2 = |A(z, T)\psi(r, \theta)|^2 = |A(z, T)|^2|\psi(r, \theta)|^2$ , Eq. (2-7) reduces to

$$j\frac{\partial A(z, T)}{\partial z} + \frac{1}{2}k''\frac{\partial^2 A(z, T)}{\partial T^2} - \gamma |A(z, T)|^2 A(z, T) = 0, \quad (2-10)$$

where

$$\gamma = \frac{\omega_0^2 \mu_0 \epsilon_0}{k^{(0)}} \frac{\int dS n_0 n_2 |\psi|^4}{\int dS |\psi|^2}. \quad (2-11)$$

Equation (2-10) is the *nonlinear Schrödinger* (NLS) equation. More on the equation follows in the next section. Note that the equation has the same form as Eq. (2-22) in Chap. 4, which describes the propagation of beams in a cubically nonlinear medium.

In closing this section we point out that under certain conditions, the two fiber parameters, GVD and SPM, can cooperate in such a way as to support fiber soliton transmission. Fiber solitons and their relationship to the NLS equation are discussed in the next section.

### 3 FIBER-OPTIC SOLITONS AND THE NLS EQUATION

In order to understand the physics governing fiber-optic soliton generation and their propagation inside optical fibers, it is best to begin with an analysis of the basic propagation equation that describes their motion. A discussion of solitons begins by examining the detailed pulse propagation equation and how it relates the dispersive and nonlinear components to soliton evolution. The detailed pulse propagation equation for single-mode fibers can be written as (Agrawal, 1995)

$$\begin{aligned} & \frac{\partial A}{\partial z} + \frac{\alpha}{2}A - \frac{j}{2}k''\frac{\partial^2 A}{\partial T^2} - \frac{1}{6}k''' \frac{\partial^3 A}{\partial T^3} \\ & = -j\gamma \left[ |A|^2 A - j\delta \frac{\partial}{\partial T} (|A|^2 A) - \Gamma_R A \frac{\partial |A|^2}{\partial T} \right], \end{aligned} \quad (3-1)$$

where  $\alpha$  accounts for fiber loss,  $k''$  relates to fiber dispersion (GVD), and  $k'''$  signifies the higher-order GVD slope parameter. The three terms on the right-hand side of Eq. (3-1) quantify the fiber nonlinearities. The first term is the result of SPM, the second term accounts for *self-steepening* of the optical

pulse, and the last term relates to the self-frequency shift encountered in ultrashort optical pulses because of *Raman gain*. Note that the NLS equation derived in the previous section is a simplified version of the above equation.

Although in most practical applications the higher-order terms ( $k'''$ ,  $\delta$ ,  $\Gamma_R$ ) may be ignored, there are occasions when their contribution must be included. For example, if the system is operating at (or very near) the *zero-dispersion wavelength*, then  $k'' \approx 0$  and  $k'''$  accounts for the dispersion effects. In addition, for ultrashort pulse widths ( $< 0.1$  psec) the GVD slope in general becomes important even for cases when  $k''$  is nonzero. This is because the pulse spectral width ( $\Delta\omega = 1/T_0$ ) becomes comparable to the carrier frequency  $\omega_0$ , so that  $\Delta\omega/\omega_0$  is not small enough to ignore the  $k'''$  term in Eq. (3-1)). The higher-order nonlinear term associated with self-steepening of the pulse shape is governed by the parameter  $\delta$ . This term is the result of the intensity dependence of the pulse group velocity, and in general should be included for high-intensity pulses. The last term in the detailed pulse propagation equation is related to the *Raman gain response*  $\Gamma_R$ , which may be attributed to the molecular vibrations associated with the nonlinear susceptibility (see Chap. 7). Except under special circumstances, the Raman gain response may be neglected.

In general, if the pulse width is greater than about 1 psec, some of the higher-order fiber terms become negligible, resulting in a simplified pulse propagation equation, given by

$$j \frac{\partial A}{\partial z} + j \frac{\alpha}{2} A + \frac{1}{2} k'' \frac{\partial^2 A}{\partial T^2} - \gamma |A|^2 A = 0. \quad (3-2)$$

It is from this simplified pulse propagation equation that the standard form for the NLS equation used in the discussion below is taken. By ignoring fiber loss ( $\alpha = 0$ ), Eq. (3-2) reduces to the NLS equation we derived in Eq. (2-10). It is usually normalized by redefining the amplitude  $A$ , space  $z$ , and time  $T$  variables as follows:

$$u = \sqrt{\frac{\gamma T_0^2 A^2}{|k''|}}, \quad \xi = \frac{z}{L_D}, \quad \tau = \frac{T}{T_0}, \quad (3-3)$$

yielding the *normalized NLS equation*

$$j \frac{\partial u}{\partial \xi} + \frac{1}{2} \text{sgn}(k'') \frac{\partial^2 u}{\partial \tau^2} - |u|^2 u = 0 \quad (3-4)$$

In Eq. (3-4),  $T_0$  is the initial pulse width while  $L_D = \frac{T_0^2}{|k''|}$  is the dispersion length. In addition,  $\text{sgn}(k'')$  denotes a *signum function*, which equals  $-1$  in the anomalous dispersion regime, where solitons are expected (Agrawal, 1995).

The inclusion of dispersion alone in the simplified pulse propagation equation [Eq. (3-2)] with  $\alpha = 0$  results in the following form for the PDE and its solution:

$$j \frac{\partial A}{\partial z} = -\frac{1}{2} k'' \frac{\partial^2 A}{\partial T^2}. \quad (3-5)$$

This equation is of the same form as that of the paraxial wave equation used to describe beam propagation in linear optics [see Eq. (4-5) in Chap. 1]. Thus, a Gaussian input pulse  $A(0, T) = \exp -\frac{1}{2} \frac{T^2}{T_0^2}$  remains a Gaussian pulse and is given by

$$\begin{aligned} A(z, T) &= \frac{T_0}{\sqrt{T_0^2 + jk''z}} \exp -\frac{1}{2} \frac{T^2}{(T_0^2 + jk''z)} \\ &= |A(z, T)| e^{-j\phi_L(z, T)}, \end{aligned} \quad (3-6)$$

where the phase  $\phi_L(z, T)$  is time dependent and the instantaneous frequency  $d\omega = \frac{-\partial\phi_L}{\partial T}$  implies a linear frequency *chirp*. Therefore, an initially unchirped pulse will develop a chirp as it propagates along the fiber because of the effect of fiber dispersion.

Likewise, by including nonlinearity alone, Eq. (3-2) and its solution become

$$j \frac{\partial A}{\partial z} = \gamma |A|^2 A \quad (3-7)$$

and

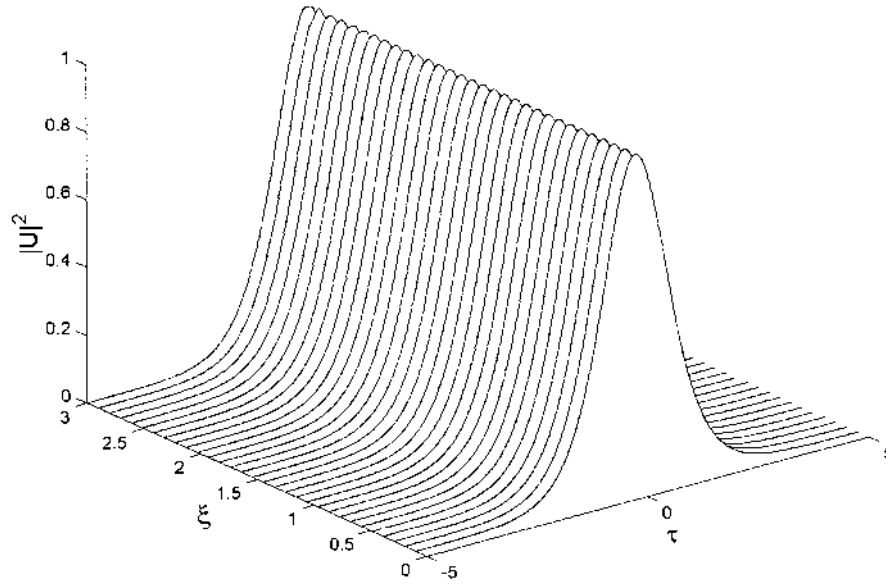
$$A(z, T) = A(0, T) e^{-j|A(0, T)|^2 z} = A(0, T) e^{-j\phi_{NL}(z, T)}, \quad (3-8)$$

where the phase  $\phi_{NL}(z, T)$  is time and intensity dependent. Again, the frequency  $d\omega = \frac{-\partial\phi_{NL}}{\partial T}$  implies a frequency chirp due to the fiber nonlinearity (Agrawal, 1995).

For a fiber-optic system operating in the A-D regime ( $k'' < 0$ ) it is not difficult to demonstrate the physical phenomenon governing soliton formation, described as follows: The GVD-induced chirp decreases (negative slope) from the leading edge of the pulse to the trailing edge. The SPM-induced chirp increases (positive slope) from the leading edge to the trailing edge of the pulse. The two chirps essentially cancel one another, allowing the pulse to propagate without change in either its amplitude or its spectrum (Korpel and Banerjee, 1984).

The *fundamental soliton solution* of Eq. (3-4) for  $k'' < 0$  can be determined using the same technique as that used to find the solution for spatial solitons in one transverse dimension, discussed in Sec. 2 of Chap. 4. The result is (Hasegawa and Tappert, 1973).

$$u(\tau, \xi) = \text{sech}(\tau) \exp - (j\xi/2). \quad (3-9)$$



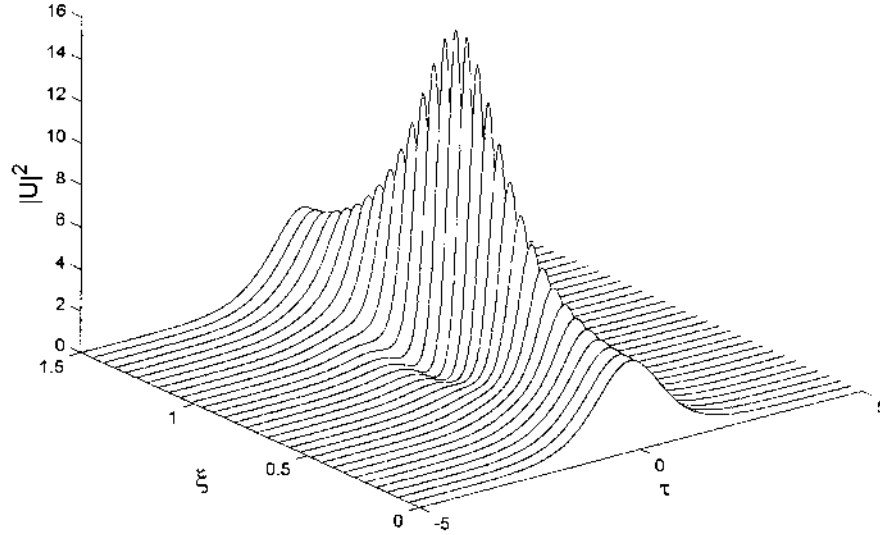
**Figure 2** Propagation of first order or fundamental soliton.

We show the shape of the pulse in Fig. 2 by numerically propagating a pulse  $u(\iota, 0) = \text{sech}(\iota)$  using Eq. (3-4) and using a so-called *fully adaptive wavelet transform* (FAWT) technique. A full detailed description of the FAWT technique is outside the scope of this book, but we outline the essentials of wavelet transforms, multiresolution analysis, and its application to the NLS equation in Appendix B. The problem can also be numerically solved using well-known split-step Fourier methods (see Appendix A).

Numerical results for a *second-order soliton* pulse are shown in Fig. 3. It shows the intensity profile of the hyperbolic secant pulse for one period of the second-order soliton. For a second-order soliton, the initial amplitude of the hyperbolic secant pulse is scaled by a factor of two.

#### 4 DISPERSION MANAGED SOLITON COMMUNICATION SYSTEMS

This is a rather recent area of interest in the area of *fiber-optic communication* and potentially is the answer to ultrahigh data communication rates over very long distances. Data communication channels of 40 Gbits/sec over 10,000 km are currently achievable, and experiments have recently demonstrated a 1000 Gbits/sec per 1000 km laboratory-grade communication



**Figure 3** Propagation of a second order soliton. Note the periodic behavior as a function of propagation distance.

channel (Hasegawa, 2000). Thus, the importance of *dispersion managed soliton systems* (DMSS) is quite clear. In DMSS, the dispersion characteristic of the fiber is profiled along the direction of propagation. Typical profile shapes are shown later during simulation of pulses through such a system. Essentially, the group velocity dispersion is periodic in nature, with positive and negative values of GVD over the repetition period. The particular form of the DMSS equation used in this research is a slight variation of the traditional NLS equation, with slightly modified dispersion and nonlinear terms and an additional term to account for fiber loss. The DMSS equation may be given as

$$j \frac{\partial u}{\partial \xi} + j \frac{1}{2} \alpha(\xi) u + \frac{1}{2} k''(\xi) \frac{\partial^2 u}{\partial \tau^2} + \gamma_0 |u|^2 u = 0. \quad (4-1)$$

In the DMSS equation  $\alpha(\xi)$  represents the fiber loss as a function of  $\xi$ ,  $k''(\xi)$  represents fiber dispersion, which changes with  $\xi$ , and  $\gamma_0$  represents a nonlinearity coefficient.

Simulation results are presented below for two different DMSS designs using Eq. (4-1), with the following sets of fiber parameters:

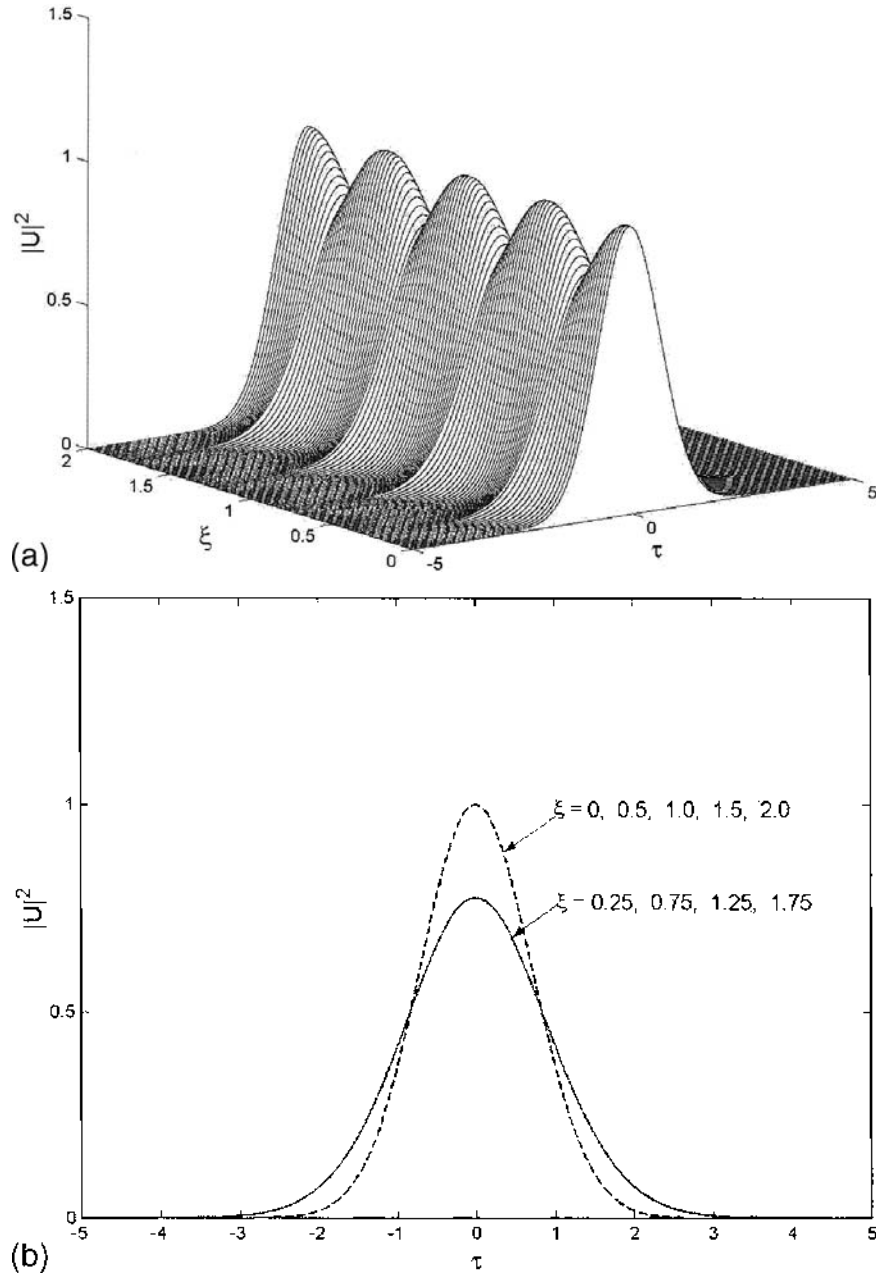
- DMSS 1: Gaussian pulse,  $k'' = [-4.0: + 3.8]$ ,  $\gamma_0 = 1$ ,  $\alpha = 0$  dB/unit of  $\xi$   
 DMSS 2: Gaussian pulse,  $k'' = [-4.0: + 3.8]$ ,  $\gamma_0 = 2$ ,  $\alpha = 0$  dB/unit of  $\xi$

The first system design is shown in Fig. 4. Figure 4a shows the propagation of the Gaussian pulse, Fig. 4b gives the intensity profile of the soliton at its maximum and minimum intensity levels, Fig. 4c portrays the relationship between the peak intensity values and the pulse widths of the pulse, and Fig. 4d displays the fiber parameters of dispersion  $k''$ , nonlinearity  $\gamma_0$ , and loss  $\alpha$ , as a function of the propagation parameter  $\xi$ . As the Gaussian pulse begins to travel through the fiber under anomalous dispersion conditions ( $k'' < 0$ ), the pulse begins to disperse, as is the case for a Gaussian pulse input to the NLS equation. However, by switching to normal dispersion ( $k'' > 0$ ) at  $\xi = 0.25$ , the effects of dispersion are negated and the pulse begins to increase in intensity and decrease in width. If this condition were to continue, however, the pulse would eventually become unstable and break apart. Switching back to anomalous dispersion once again generates pulse dispersion, so the pulse again starts to disperse and die out. By continuing this procedure repeatedly, the pulse shape is maintained for period after period of the dispersion cycle. The measured parameter  $T_{\text{FWHM}}$  is related to the commonly used half-width at  $1/e$  intensity point  $T_{1/e}$  by  $T_{\text{FWHM}} = T_{1/e} 2\sqrt{\ln 2}$ . As indicated in these four figures, DMSS 1 is a “balanced” system, in which fiber parameters are chosen in such a way as to create a useful *optical soliton communication system*.

By simply doubling the nonlinear coefficient, another unique system is created in DMSS 2 as shown in Fig. 5. The same parameters are plotted in Fig. 5a through d as was done for DMSS 1. This scheme can also be considered a soliton communication system. However, it might encounter problems in optical communication receiver design by having “two” peak intensity levels instead of just one. In addition, the rapid changes in peak intensity and pulse width levels (Fig. 5c) may generate energy dispersion problems if the fiber parameters are not kept within strict tolerance levels.

## 5 DARK SOLITONS

Thus far, we have investigated the propagation of solitons in an optical fiber in the anomalous dispersion regime, and also studied propagation of solitons in dispersion managed optical fibers where the dispersion switches from anomalous to normal in a periodic fashion. The solitons that exist in the anomalous dispersion regime are called *bright solitons*. On the other hand, an optical fiber, under normal dispersion regime, supports the so-called *dark solitons*, where the amplitude of the complex electric field envelope  $u$  tends to a constant for the retarded time  $\tau$ . It has been found that dark solitons, when compared to bright ones, decay less in the presence of damping (Kivshar and Yang, 1994) and are less susceptible to *jitter* (*Gordon–Haus effect*, for an



**Figure 4** (a) Pulse propagation in DMSS #1. (b) Intensity Profile in DMSS #1. (c) Soliton intensity and pulse width for DMSS #1. (d) Optical fiber parameters used for DMSS #1. In (d),  $\beta_2$  has the same connotation as  $k''$ .

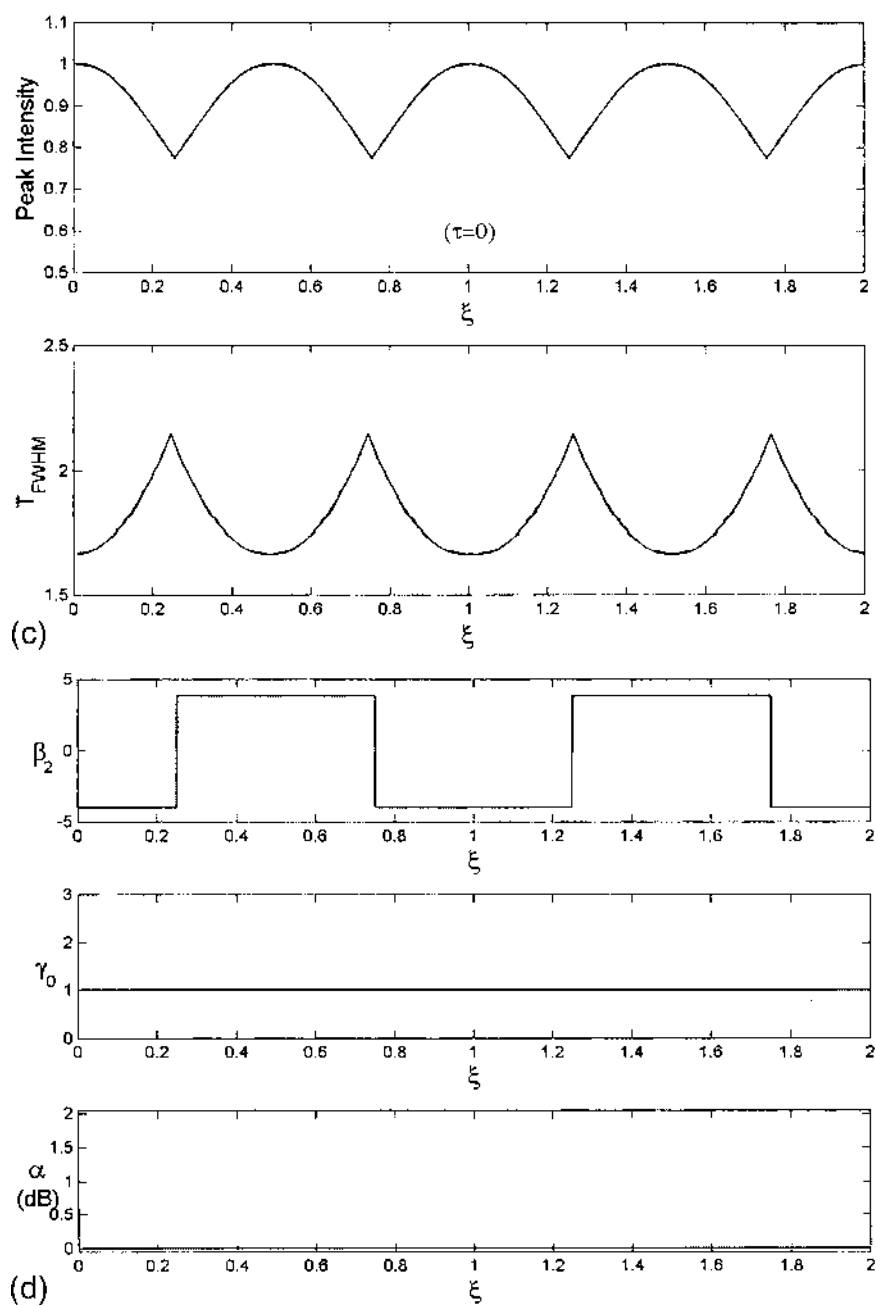
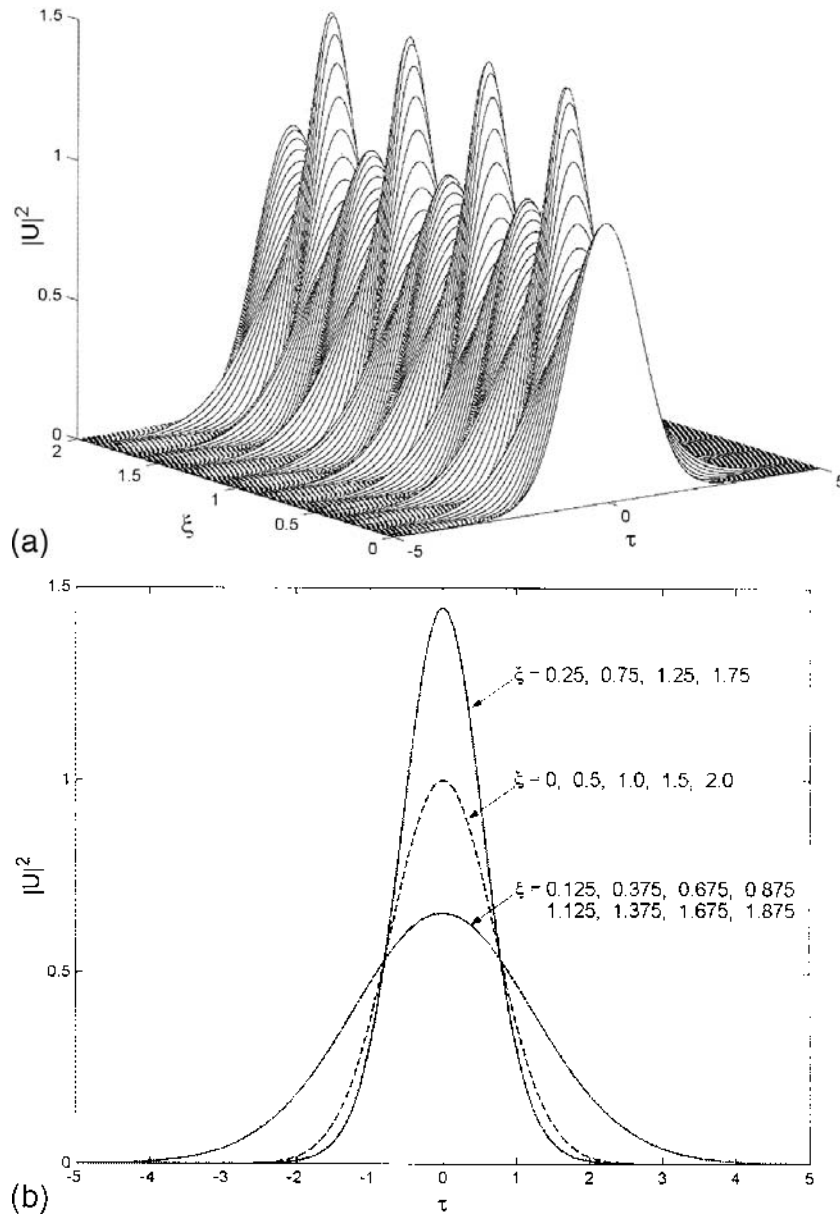


Figure 4 Continued.



**Figure 5** (a) Pulse propagation in DMSS #2. (b) Intensity profile in DMSS #2. (c) Soliton intensity and pulse width for DMSS #2. (d) Optical fiber parameters used for DMSS #2.

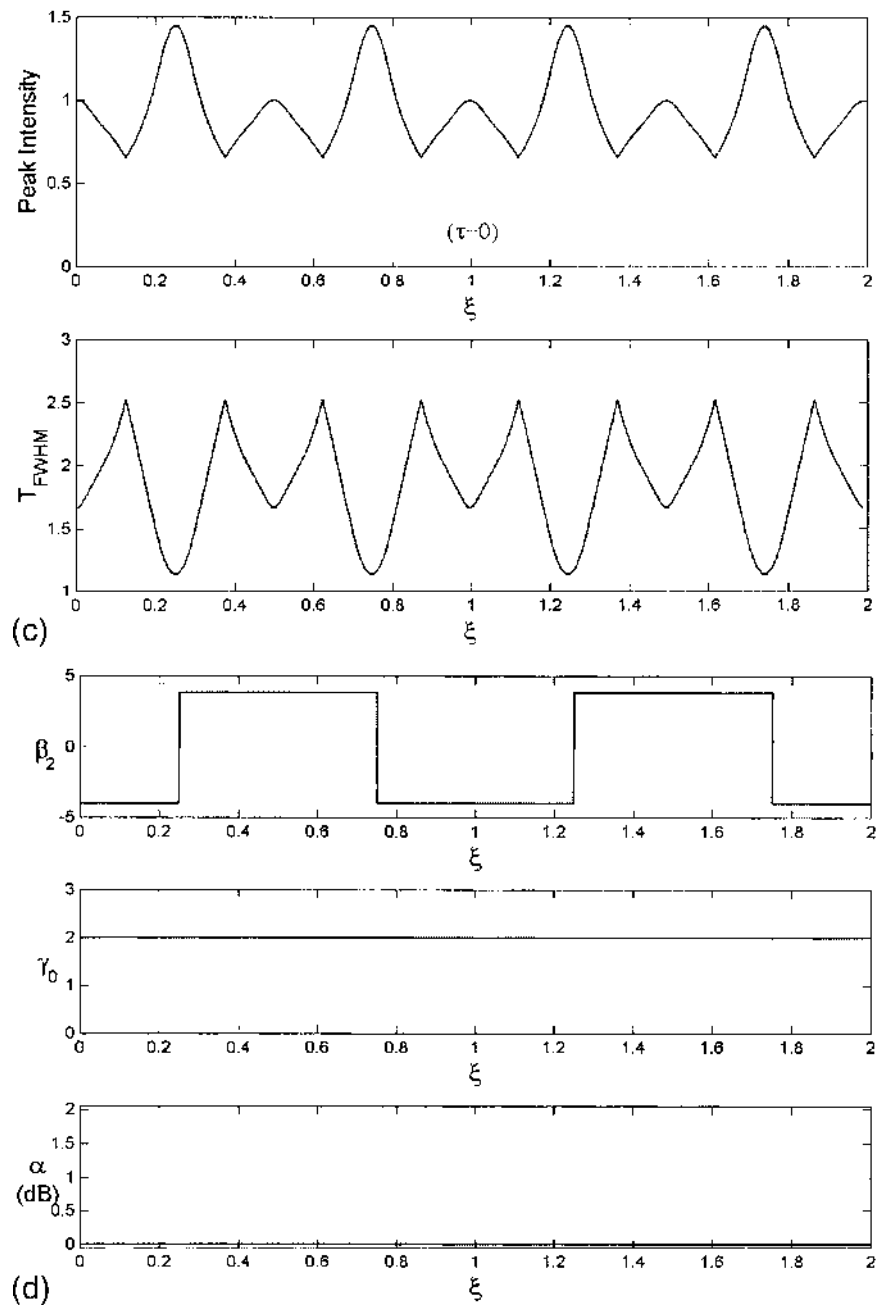


Figure 5 Continued.

explanation of this please refer to Gordon and Haus, 1986; Kivshar et al., 1994). Because of these properties, dark solitons may actually be preferable to bright ones when used as bit carriers in fiber-optics communication systems. Since the first successful dark soliton generation experiment (Emplit et al., 1987), effective methods to generate high-bit-rate dark soliton trains (Richardson et al., 1994) have become available. In fact, the first successful experiment of 10 Gbits/sec pseudorandom dark soliton data transmission over 1200 km was recently reported in Nakazawa and Suzuki (1995). In order to assess the effectiveness of dark pulses for longer communication distances, it is important to examine their dynamical behavior in the presence of external perturbations in the same detail as for bright soliton propagation (Burtsev and Camassa, 1997).

In the normal dispersion regime, the NLS equation [Eq. (3-4)] becomes

$$j \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} - |u|^2 u = 0. \quad (5-1)$$

The dark soliton solution of Eq. (5-1) is

$$u = \exp(-ju_0^2 \xi) [1/v + j\sigma \tanh \sigma(\tau + \xi/v)]. \quad (5-2)$$

Here  $u_0$  is the background amplitude,  $\sigma$  is the dark soliton amplitude, and  $v$  is the velocity. Of the three free parameters  $u_0$ ,  $\sigma$ , and  $v$ , only two are arbitrary since the relation

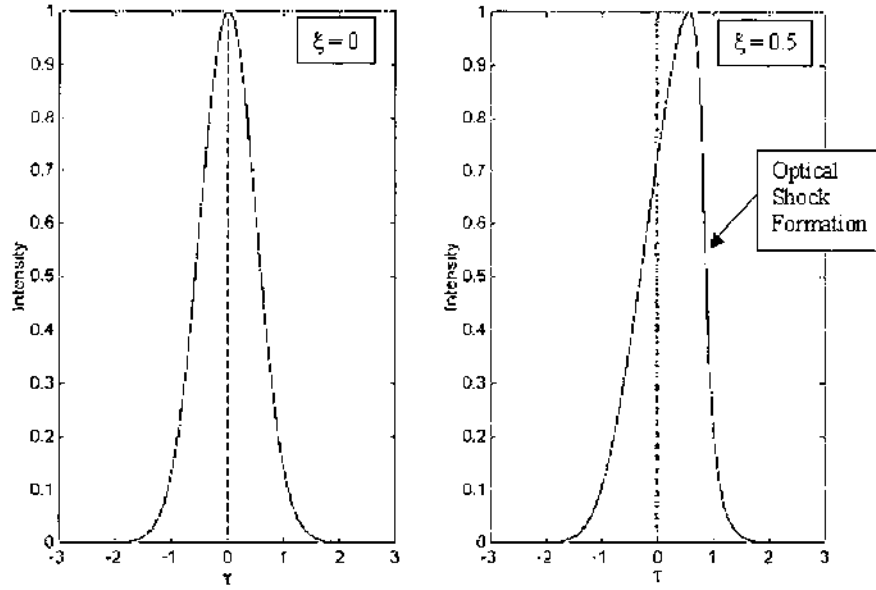
$$u_0^2 = \sigma^2 + 1/v^2 \quad (5-3)$$

must be satisfied for Eq. (5-2) to be a solution of Eq. (5-1).

For work on dark spatial solitons, the reader is referred to Allan et al. (1991).

## 6 OPTICAL SHOCKS AND SELF-STEEPENING OF PULSES

Oliveira and Moura (1998) and Zaspel (1999) are among the many who have recently published excellent papers describing not only the processes of self-steepening and *optical shock formation*, but also the analytic solutions to these equations. Equation (3-1) included the effects of optical pulse *self-steepening*. The physical process governing self-steepening is related to the fact that the group velocity parameter may be a function of not only frequency, but also the intensity. This intensity dependence of the group velocity parameter must be included in the case of very short optical pulses whose pulse width is less than 100 fsec, as described in the previous section (Agrawal, 1995). Self-steepening will lead to an asymmetrical steepening, or *shock wave formation*,



**Figure 6** Self-steepening of an optical pulse for  $S = 0.5$ .

of the trailing edge of the optical pulse, as indicated in Fig. 6. The effect of GVD on the process of steepening is a gradual dissipation in the shock front and eventual reduction, or broadening, of the steepened trailing edge. However, by considering only the effects of nonlinearity and self-steepening, and ignoring fiber dispersion and all other higher-order effects, the result becomes the basic *self-steepening equation*:

$$j \frac{\partial u}{\partial \xi} + jS \frac{\partial}{\partial \tau} (|u|^2 u) - |u|^2 u = 0 \quad (6-1)$$

where  $S$  governs the amount of self-steepening experienced by the pulse during propagation along the fiber. If the self-steepening parameter  $S$  is positive, the pulse will develop a shock front in the trailing edge, i.e., will “steepen right.” If  $S$  is negative, the pulse will develop a shock in the leading edge or “steepen left.” For the results shown in Fig. 6,  $S = 0.5$ . Equation (6-1), which ignores fiber dispersion, is used to numerically investigate pulse propagation in this case, along with the FAWT technique, in order to resolve steep gradients associated with self-steepening.

In closing, we remark that self-steepening leading to optical gives rise to very large gradients in the slope of the pulse as it propagates through the

optical medium. Numerically, analysis of this can cause instabilities. The primary advantage offered by wavelet analysis used to simulate the results shown above compared to other techniques is the highly adaptive nature of wavelets. By switching to higher and higher wavelet levels the FAWT is able to accurately track and resolve these steep gradients.

## 7 PROBLEMS

1. Starting from the NLS equation, show that there is conservation of energy during propagation of the pulse.
2. Using the beam propagation method (Appendix A, Poon and Banerjee, 2001), simulate the propagation of a pulse in a medium with positive and negative values of the nonlinearity and positive and negative values of the group velocity dispersion parameter.
3. Derive the evolution equation for envelopes that are localized in space and time ( $3 + 1$  dimensions) propagating in a nonlinear dispersive medium. Assuming radial symmetry of these envelopes in space, reduce the equation to one involving time and the radial coordinate  $R$ . (Hint: You may like to refer to Korpel and Banerjee, 1984).
4. Solitons in an optical fiber operating at 1.55 microns have a pulse duration of 4 psec. The dispersion coefficient is given to be 16 psec/nm km. Assume the refractive index to be 1.45, and the nonlinear refractive index coefficient to be  $3.2 \times 10^{-20} \text{ m}^2/\text{V}^2$ . Calculate the peak amplitude of the soliton.
5. Starting from the NLS equation in the normal dispersion regime, derive the analytic expression for a one-dimensional dark soliton [Eq. (5-2)].

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# 9

## Photorefractive Nonlinear Optics

### 1 INTRODUCTION

The *photorefractive* (PR) effect refers to the light-induced change of refractive index in an electro-optic optically sensitive doped semiconductor. Since its first discovery by Ashkin et al. (1966), a tremendous amount of research has been carried out to study the PR effect and apply it to real-time image processing, beam amplification, self-pumped phase conjugation, four-wave mixing, optical computing, etc. (Yeh, 1993). When two coherent plane waves of light intersect in a PR material, they form an intensity interference pattern comprising bright and dark regions. In a PR material that is predominantly n-doped, electrons migrate from bright to dark regions, thus creating an approximately sinusoidal charge distribution. This diffusion-controlled PR effect in turn creates an *electrostatic space charge field* which is ideally phase-shifted from the intensity pattern by  $90^\circ$  and modulates the refractive index of the material via the electro-optic effect. A *grating* is thus induced in the PR material. The incident plane waves are, in turn, scattered by the grating in a way that one wave may have constructive recombination, while the other may encounter a destructive recombination. This effect leads to energy coupling between the beams through what is commonly referred to as the *two-beam coupling effect* (Yeh, 1993).

In this chapter, in Sec. 2, we first introduce readers to the *Kukhtarev equations* (Kukhtarev et al., 1979) that model charge generation and transport in a PR medium, which in turn induces the electrostatic field. We next reduce these equations to a set of two exact coupled nonlinear differential equations which then need to be solved in conjunction with *Maxwell's equations* in the material. Furthermore, in order to gain more physical insight, a simplified differential equation for the electrostatic field (and hence the induced refractive index) is derived.

The steady-state solution found from the simplified differential equation is then used in Sec. 3, along with the *beam propagation* algorithm, to

study *beam fanning* and two-wave mixing in PR materials. Using the steady-state solution for the electrostatic field, we numerically study (Gaussian) beam fanning in thin and thick PR materials. It turns out that only for the thin sample case can an analytical solution be found, which is useful when comparing with numerical simulations. Results of the thin sample analysis show that the beam distorts in the far field. The distortion is due to the self-induced change in the refractive index during propagation through the PR material. We examine beam distortion in photorefractive materials such as *barium titanate* (where diffusion effects dominate) and in *lithium niobate* (where photovoltaic effects dominate). Having examined the propagation of a single Gaussian beam, we numerically study the more complicated case of coupling between two focused Gaussian beams. The numerical simulations of this case can be reconciled with the steady-state nonlinear theory of the two-beam coupling phenomenon, using participating plane waves, as shown by Ratnam and Banerjee (1994). The results indicate that the *two-beam coupling parameter* (Yeh, 1993) strongly depends on the initial intensity ratio of the plane waves (or power ratio of the input beams).

In Sec. 4, we are concerned with solving the optical equations, along with the approximate solution for the electrostatic space charge field and hence the induced refractive index, in the steady state, to study two-wave mixing in PR materials. The interaction of optical waves in a PR material may induce a transversely periodic refractive index profile (as in a *transmission grating*) or a longitudinally varying refractive index profile (as in a *reflection grating*). In reality, the PR material may be mismatched with respect to the refractive index of the surrounding medium. The analysis of this case, along with the temporal interaction of the interacting optical waves, can be handled by a method called the *rigorous coupled wave analysis* (RCWA) and is outside the scope of this book. Interested readers are referred to Jarem and Banerjee (2000) and references therein. The RCWA algorithm, which can solve diffraction from arbitrary gratings, provides a powerful tool to study diffraction from the induced gratings leading to beam coupling and higher diffraction order generation, which is used to study diffraction from gratings.

In Sec. 5, we discuss phase conjugation using PR materials. First, we describe phase conjugation using a traditional *degenerate four-wave mixing* (DFWM) geometry, as discussed in Chap. 6. Thereafter, we describe *self-pumped phase conjugation* (SPPC) as a special case of the four-wave mixing principle. Other applications of PR materials to image processing, such as self-phase conjugation and edge enhancement, are introduced in Sec. 6.

Finally, in Sec. 7, we discuss a rather new class of PR materials, namely, *photorefractive polymers*. These are relatively cheap, can have high gains (depending on the applied voltage), and can also be used for two-wave coupling, higher-order generation, and image processing.

## 2 THE KUKHTAREV EQUATIONS AND THEIR SIMPLIFICATION

In this section, we first introduce readers to the *Kukhtarev equations* that model charge generation and transport in a PR medium, which in turn induces the electrostatic field. We next reduce these equations to a set of two exact coupled nonlinear differential equations which then need to be solved in conjunction with Maxwell's equations in the material. Furthermore, in order to gain more physical insight, a simplified differential equation for the electrostatic field (and hence, the induced refractive index) is derived.

The time-dependent Kukhtarev equations for a *diffusion-dominated* PR material assuming transverse field components ( $\mathbf{J} = J\mathbf{a}_x$ ,  $\mathbf{E}_s = E_s\mathbf{a}_x$ ) and transverse coordinate  $x$  are (Yeh, 1993):

$$\frac{\partial(N_D^+ - \bar{n}_e)}{\partial t} = -\frac{1}{e} \frac{\partial J}{\partial x}, \quad (2-1)$$

$$\frac{\partial N_D^+}{\partial t} = (N_D - N_D^+)(sI + \beta) - \gamma_R N_D^+ \bar{n}_e, \quad (2-2)$$

$$J = eD_s \frac{\partial \bar{n}_e}{\partial x} + e\mu_e n_e E_s, \quad (2-3)$$

$$\epsilon_s \frac{\partial E_s}{\partial x} = e(N_D^+ - N_A - \bar{n}_e), \quad (2-4)$$

where  $\bar{n}_e$  is the *free electron density*,  $N_D^+$  is the *ionized donor concentration*,  $J$  is the *current density*,  $E_s$  is the *electrostatic space charge field*, and  $I$  is the intensity distribution of the optical field. The *donor concentration* is denoted by  $N_D$ , and the *acceptor concentration* by  $N_A$ . Both are assumed constant with respect to time and space. Furthermore,  $e$  is the electronic charge,  $s$  is the *ionization cross section*,  $\beta$  is the *thermal excitation rate* (proportional to *dark current*),  $\gamma_R$  is the *recombination rate*,  $D_s$  is the *diffusion constant*,  $\epsilon_s$  is the *effective quasi-static permittivity*, and  $\mu_e$  is the *effective quasi-static mobility*. The effective values of  $\epsilon_s$  and  $\mu_e$  are defined by Yeh (1993).

The four Kukhtarev equations written above can be reduced, without any simplifications, to a set of two coupled nonlinear equations involving two dependent variables. To this end, we substitute Eq. (2-3) into Eq. (2-1) and find

$$\frac{\partial \bar{n}_e}{\partial t} = \frac{\partial N_D^+}{\partial t} + \mu_e \frac{\partial \bar{n}_e E_s}{\partial x} + D_s \frac{\partial^2 \bar{n}_e}{\partial x^2}. \quad (2-5)$$

Using Eq. (2.4), we find

$$N_D^+ = \frac{\epsilon_s}{e} \frac{\partial E_s}{\partial x} + N_A + \bar{n}_e, \quad (2-6)$$

or

$$\frac{\partial N_D^+}{\partial t} = \frac{\epsilon_s}{e} \frac{\partial^2 E_s}{\partial x \partial t} + \frac{\partial \bar{n}_e}{\partial t}, \quad (2-7)$$

where the fact that  $\frac{\partial N_A}{\partial t} = 0$  has been used. If  $\frac{\partial N_D^+}{\partial t}$  from Eq. (2-7) is substituted into Eq. (2-5) and  $\frac{\partial \bar{n}_e}{\partial t}$  is canceled on the right- and the left-hand sides of the resulting equation, we obtain

$$\frac{\epsilon_s}{e} \frac{\partial^2 E_s}{\partial x \partial t} + \mu_e \frac{\partial(\bar{n}_e E_s)}{\partial x} + D_s \frac{\partial^2 \bar{n}_e}{\partial x^2} = 0. \quad (2-8)$$

Integration with respect to  $x$  then yields

$$\frac{\partial E_s}{\partial t} + \frac{\mu_e e}{\epsilon_s} \bar{n}_e E_s = -D_s \frac{e}{\epsilon_s} \frac{\partial \bar{n}_e}{\partial x}. \quad (2-9)$$

In this equation, we have set the integration constant equal to zero, assuming that all dependent variables and their derivatives tend to zero as  $|x| \rightarrow \infty$ . This is a good assumption because in reality, all optical beams and optical materials have finite size. It is useful to define a dimensionless quantity called the induced dielectric permittivity modulation  $\Delta\epsilon$ . Since we will be interested mainly in the analysis of optical propagation through BaTiO<sub>3</sub>, it is proportional to the electrostatic field  $E_s$  and is given by

$$\Delta\epsilon = n_o^2 n_e^2 r_{42} E_s, \quad (2-10)$$

where  $n_o$  and  $n_e$  represent the ordinary and extraordinary refractive indices of the PR material (crystal), respectively, and  $r_{42}$  denotes an electro-optic coefficient of BaTiO<sub>3</sub>. Substituting  $E_s$  from Eq. (2-10), we find the following equation for  $\Delta\epsilon$ :

$$\frac{\partial \Delta\epsilon}{\partial t} + \frac{\mu_e e}{\epsilon_s} (\bar{n}_e \Delta\epsilon) = \frac{D_s e}{\epsilon_s} r_{42} n_o^2 n_e^2 \frac{\partial \bar{n}_e}{\partial x}. \quad (2-11)$$

Up to this point, no use has been made of Eq. (2-5), the electron rate production equation. If  $\frac{\partial N_D^+}{\partial t}$  from Eq. (2-7) is substituted in Eq. (2-5), we find

$$\frac{\epsilon_s}{e} \frac{\partial^2 E_s}{\partial x \partial t} + \frac{\partial \bar{n}_e}{\partial t} = (sI + \beta)(N_D - N_D^+) - \gamma_R \bar{n}_e N_D^+. \quad (2-12)$$

When Eq. (2-8) is used to eliminate  $\frac{\epsilon_s}{e} \frac{\partial^2 E_s}{\partial x \partial t}$ , and  $N_D^+$  of Eq. (2-6) is substituted above, we find

$$\begin{aligned} -\mu_e \frac{\partial(\bar{n}_e E_s)}{\partial x} - D_s \frac{\partial^2 \bar{n}_e}{\partial x^2} + \frac{\partial \bar{n}_e}{\partial t} &= (sI + \beta) \left( N_D - \frac{\epsilon_s}{e} \frac{\partial E_s}{\partial x} - N_A - \bar{n}_e \right) \\ &\quad - \gamma_R \bar{n}_e \left( \frac{\epsilon_s}{e} \frac{\partial E_s}{\partial x} + N_A + \bar{n}_e \right). \end{aligned} \quad (2-13)$$

At this point, it is useful to introduce normalized coordinates and variables. Letting  $C = \beta/s$ ,  $(sI + \beta) = sC(1 + I/C)$ ,  $x' = k_0x$ ,  $t' = \beta t$ ,  $\tilde{n} = \bar{n}_e/N_A$ , using Eq. (2-10) to express  $E_s$  in terms of  $\Delta\epsilon$ , and dropping the primes on  $x$  and  $t$  for convenience, we obtain

$$\begin{aligned} & \Gamma_2 \frac{\partial^2 \tilde{n}}{\partial x^2} + \Gamma_1 \Delta\epsilon \frac{\partial \tilde{n}}{\partial x} \\ & + \left\{ -\Gamma_3 \left( 1 + \frac{I}{C} \right) + (\Gamma_1 - \Gamma_4) \frac{\partial \Delta\epsilon}{\partial x} - \Gamma_3 \Gamma_4 (1 + \tilde{n}) \right\} \tilde{n}, \\ & = \Gamma_3 \left\{ \frac{\partial \tilde{n}}{\partial t} - \left( 1 + \frac{I}{C} \right) \left( \frac{N_D}{N_A} - 1 - \frac{1}{\Gamma_3} \frac{\partial \Delta\epsilon}{\partial x} \right) \right\} \end{aligned} \quad (2-14)$$

where

$$\Gamma_1 = \frac{\mu_e e N_A}{\beta \epsilon_s}, \Gamma_2 = \frac{ek_0 D_s N_A n_0^2 n_e^2 r_{42}}{\beta \epsilon_s}, \Gamma_3 = \frac{e N_A n_0^2 n_e^2 r_{42}}{\epsilon_s k_0} \quad (2-15)$$

$$\Gamma_4 = \frac{\gamma_R N_A}{\beta}$$

Furthermore, Eq. (2-11) in normalized form can be written as

$$\frac{\partial \Delta\epsilon}{\partial t} + \Gamma_1 \tilde{n} \Delta\epsilon = -\Gamma_2 \frac{\partial \tilde{n}}{\partial x}. \quad (2-16)$$

Equations (2-14) and (2-16) are a pair of coupled, nonlinear equations for the electron density  $\tilde{n}$  and the induced dielectric modulation function  $\Delta\epsilon$ . The form of both of these equations for  $\tilde{n}$  and  $\Delta\epsilon$  at any given time and at any given point in the PR material depends on the value of the optical intensity  $I$  at that point in space and time. The value of  $I$  itself in the PR material depends on the incident optical field as well as on the optical energy which has been transmitted, reflected, and diffracted by the dielectric modulation function  $\Delta\epsilon$  that exists in the PR material at a given time.

The model above has been limited to one transverse dimension. A more complete model, including transverse as well as longitudinal electrostatic field components in the Kukhtarev equations, as well as longitudinal variations of the electrostatic fields, has been developed (Jarem and Banerjee, 1996) but will be omitted from our discussion for the sake of simplicity.

To gain a physical insight into the generation of the induced refractive index profile, it is instructive to further simplify the foregoing set of two coupled equations. This can be accomplished by using the following approximations, involving the electron rate production equation. Numerical evaluation of each of the terms in Eq. (2-14) is invaluable in determining which terms have the most and least significance. The numerical

experimentation shows that a valid zero-order approximation of Eq. (2-14) reads

$$\{-\Gamma_3\Gamma_4(1+\tilde{n})\}\tilde{n} = \Gamma_3\left\{-\left(1+\frac{I}{C}\right)\left(\frac{N_D}{N_A}-1\right)\right\}. \quad (2-17)$$

Next, we assume that the donor density  $N_D$  is much greater than the acceptor density  $N_A$  and that  $\tilde{n} \equiv \bar{n}_e/N_A \ll 1$ . Using these assumptions and straight forward algebra, we find

$$\bar{n}_e = \frac{(sI + \beta)N_D}{\gamma_R N_A}. \quad (2-18)$$

Thus, the electron concentration is linearly related to the optical intensity, as expected. Finally, upon differentiating Eq. (2-18) with respect to  $x$ , substituting into Eq. (2-16), and denormalizing, we get

$$\tau(I) \frac{\partial \Delta \varepsilon}{\partial t} = -\Delta \varepsilon + \frac{m_0^4 D_s}{\mu_e (I + \beta/s)} \frac{\partial I}{\partial x} \quad (2-19)$$

where the intensity-dependent time constant can be expressed as (Jarem and Banerjee, 1996)

$$\tau(I) = \frac{\gamma_R N_A \varepsilon_s}{\mu_e e N_D (sI + \beta)}. \quad (2-20)$$

The net effect of invoking these approximations is that the two coupled equations are reduced to a single equation, which is conducive to a better physical insight. Equation (2-19) is a first-order differential equation in time, with a time constant that is inversely dependent on the intensity. Physically, this is very reasonable since one expects a faster PR response to a larger intensity. The time constant is known to be a sublinear function of the intensity, i.e.,  $\tau \propto (I)^{-\gamma}$ ,  $0 < \gamma < 1$  (Singh et al., 1997). A similar sublinear response naturally follows from our analysis as seen in Eq. (2-20).

It should be pointed out that neglecting the first- and higher-order spatial derivatives in Eq. (2-14) limits the analysis to cases where changes in the electrostatic field, the electron density, etc. in the transverse direction (i.e., along  $x$ ) are small. Thus, for example, if higher-order diffraction (which varies rapidly in the transverse direction) should be excited by the PR system, its effect cannot, in general, be studied for arbitrary spatial variations in the induced dielectric permittivity modulation (i.e., corresponding to arbitrary grating wave numbers) because the second-order spatial derivative terms may contribute more to the analysis than the first-order spatial derivatives. Thus, the analysis could at best be only valid over a specific range of grating wave numbers.

### 3 BEAM FANNING AND DISTORTION IN PHOTOREFRACTIVE MATERIALS

The phenomenon of PR beam fanning, where the incident light beam is deflected and/or distorted during propagation through a high-gain PR crystal, has been observed in BaTiO<sub>3</sub>, LiNbO<sub>3</sub>, and strontium barium niobate (SBN) (Feinberg, 1982; Banerjee and Misra, 1993). The deflection and distortion can be explained through the fact that a symmetric beam may create an asymmetric refractive index profile  $\Delta n$ . This is because  $\Delta n$  is proportional to  $\Delta \epsilon$ , which in turn is proportional to  $\nabla I$ , as evident from Eq. (2-19). This leads to beam distortion, which we call *deterministic beam fanning* (DBF) in the far field (Banerjee and Misra, 1993). We use the word *deterministic* to describe this kind of beam fanning because the beam is distorted over a large scale by a predictable induced refractive index, as opposed to *random beam fanning*, where the beam is distorted due to random irregularities in the material or its surface (Segev et al., 1990; Gu and Yeh, 1991).

Let us now examine the steady-state DBF in a diffusion-dominated PR material. From Eq. (2-19), the denormalized electrostatic field  $E_s(x, y, z)$  can be expressed as

$$\vec{E}_s \approx \frac{k_B T}{e} \frac{\nabla I}{\beta/s + I} = E_{sx} \hat{u}_x + E_{sy} \hat{u}_y \quad (3-1)$$

if  $(k_B T/e)/W^2 \ll eN_A/\epsilon_s$ , where  $W$  is the characteristic beam width of the complex envelope  $E_c(x, y, z)$  of the optical field. Now, the electrostatic field  $\vec{E}_s$  induces a refractive index change  $\Delta n_E$  for extraordinary polarized (say along  $x$ , see Fig. 1) *plane waves* of light in the PR material (assumed BaTiO<sub>3</sub> from now on) through the *linear electro-optic effect*, given by (Banerjee and Misra, 1993)

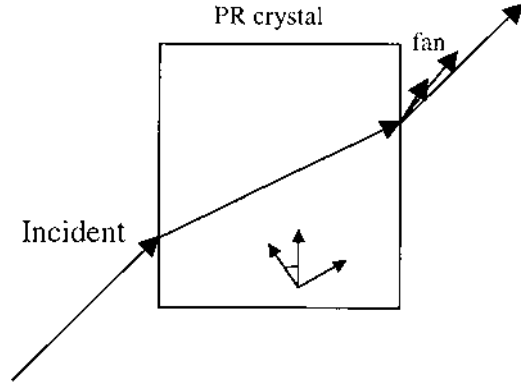
$$\Delta n_E(x, y, z, \vartheta) = E_{sx}(x, y, z) f(\vartheta), \quad (3-2a)$$

$$f(\vartheta) = -\frac{1}{2} n^3(\vartheta) \cos(\vartheta) \times [(r_{13} + 2r_{42}) \sin^2(\vartheta) + r_{33} \cos^2(\vartheta)], \quad (3-2b)$$

$$n^2(\vartheta) = [\sin^2(\vartheta)/n_o^2 + r_{33} \cos^2(\vartheta)/n_e^2]^{-1} \quad (3-2c)$$

where  $r_{ij}$  are the linear contracted electro-optic coefficients (Yeh, 1993). The angle  $\vartheta$  in Eqs. (3-2a)–(3-2c) is defined in Fig. 1. In general, propagation through the PR material under the slowly varying envelope approximation may be modeled by means of the partial differential equation (PDE)

$$\partial E_c / \partial z = -jk_0 \Delta n_E E_c - j[1/2n(\vartheta)k_0] \nabla_{\perp}^2 E_c \quad (3-3)$$



**Figure 1** Geometry used to study deterministic beam fanning in PR barium titanate. The line of sight distance from the entry face of the crystal to the exit face is  $L$ .

where

$$\Delta n_e(x, \vartheta) = \mathfrak{F}_x^{-1}[\mathfrak{F}_x[E_{sx}(x)]f(\vartheta + k_x/n(\vartheta)k_0)], \quad (3-4)$$

$$E_{sx}(x) = \frac{k_B T}{e} \frac{\partial |E_e(x)|^2 / \partial x}{\eta \beta / s + |E_e(x)|^2}$$

where  $\mathfrak{F}_x$  is a one-dimensional Fourier transform in the  $x$  coordinate and  $\eta$  is the characteristic impedance inside the PR material. The argument of  $f$  in Eq. (3-4) denotes the angle between a plane wave component of the angular plane wave spectrum of  $E_e(x)$  and the  $c$ -axis. It turns out that for values of  $\vartheta$  around  $40^\circ$ , the electro-optic effect is maximum and thus causes the greatest change in the induced refractive index. A symmetric beam will induce an asymmetric refractive index profile, leading to pronounced beam bending and DBF in the far field for  $\vartheta \approx 40^\circ$ . Equations (3-3) and (3-4) with Eqs. (3-2b) and (3-2c) form a set of coupled equations which need to be numerically solved to describe propagation of a beam through the PR material. Equation (3-1) takes into account the effect of photorefractivity, suitably modified to include the effect of  $f$  on the spatial frequency variable  $k_x$ .

A simple physical explanation of beam bending and DBF in the far field is based on the examination of the spectrum of the phase modulation  $\exp[-jk_0 \Delta n_e(x)L]$ , assuming a thin sample, in the sense that we neglect the effects of propagational diffraction through the material. As the input field immediately in front of the PR material, we assume a Gaussian profile

$$E_e(x, y, 0) = (I_0 \eta)^{1/2} \exp[-(x^2 + y^2)/W^2] \quad (3-5)$$

with  $I_o = 2P/\pi W^2$ , where  $I_o$  denotes the on-axis intensity,  $P$  is the beam power, and  $W$  is the waist size. This field is phase-modulated due to the induced refractive index profile in the PR material. The resulting field immediately behind the sample is  $E_e(x, y, L) = E_e(x, y, 0) \exp[-jk_0 \Delta n_e(x)L]$ , where  $L$  is the thickness of the PR material. The far-field pattern is the Fourier transform of the output optical field immediately behind the PR sample. Now, Fraunhofer diffraction theory (see Chap.1) states that the far field is the (scaled) Fourier transform of the field behind the PR material. Hence, using the convolution property of Fourier transforms, the far field is the convolution of:

- (a) The spectrum of the phase modulation  $\exp[-jk_0 \Delta n_e(x)L]$ .
- (b) The spectrum of the Gaussian profile  $E_e(x, y, 0)$  immediately before the PR sample.

Since  $\Delta n_e(x)$  is an odd function of  $x$  [see Eqs. (3-1) and (3-2)], it can be expanded in a power series of the form  $ax^3 - bx$ , where  $a$  and  $b$  are given by

$$b = \frac{4f(\vartheta)k_B T / e W^2}{\beta / s I_o + 1}, \quad a = \frac{(2b/W^2)(\beta / s I_o)}{\beta / s I_o + 1}. \quad (3-6)$$

Note that the coefficients of this expansion hold for all values of the ratio  $\beta / s I_o$ . The spectrum of  $\exp[-jk_0 \Delta n_e(x)L]$  is then

$$\mathfrak{F}_x\{\exp[-jk_0 \Delta n_e(x)L]\} = \frac{2\pi}{(3a)^{1/3}} Ai \frac{k_x - bk_0 L}{(3a)^{1/3}}. \quad (3-7)$$

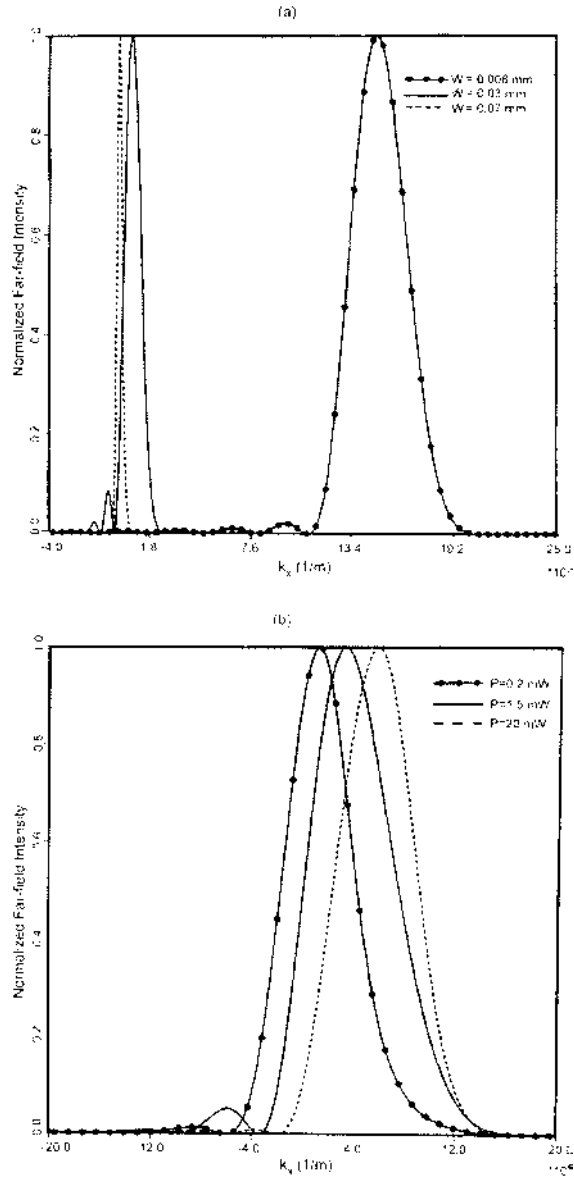
The resulting far-field pattern is therefore the convolution of the Gaussian spectrum and the *Airy pattern*. The *Airy function* resembles an asymmetric shifted damped sinusoid. Convolution of this with a Gaussian (note that the Fourier transform of a Gaussian is a Gaussian) yields a distorted Gaussian whose center is displaced from  $z=0$  and may have asymmetric sidebands, depending on the relative widths of the Airy function and the Gaussian. The shift of the far-field pattern with respect to the axis ( $z$ ) of propagation of the optical beam, and the appearance of asymmetric sidelobes, comprise the phenomenon referred to as DBF. This is in agreement with our numerical simulations below.

In what follows, we now provide exact numerical results for the far-field beam profiles for a thin PR material. By *thin* we mean that the effects of propagational diffraction through the material are neglected. Numerical simulations for BaTiO<sub>3</sub> are performed with parameters  $n_0 = 2.488$ ,  $n_e = 2.434$ ,  $r_{42} = 1640$  pm/V,  $r_{13} = 8$  pm/V,  $r_{33} = 28$  pm/V,  $N_A = 2 \times 10^{22}$  m<sup>-3</sup>,  $\epsilon_s = 3.28 \times 10^{-8}$  F/m,  $s = 2.6 \times 10^{-5}$  m<sup>2</sup>/J,  $\beta = 2$  sec<sup>-1</sup>,  $T = 298$  K (Yeh, 1993),  $L = 1$  cm, and using an incident wavelength of 514.5 nm. The results

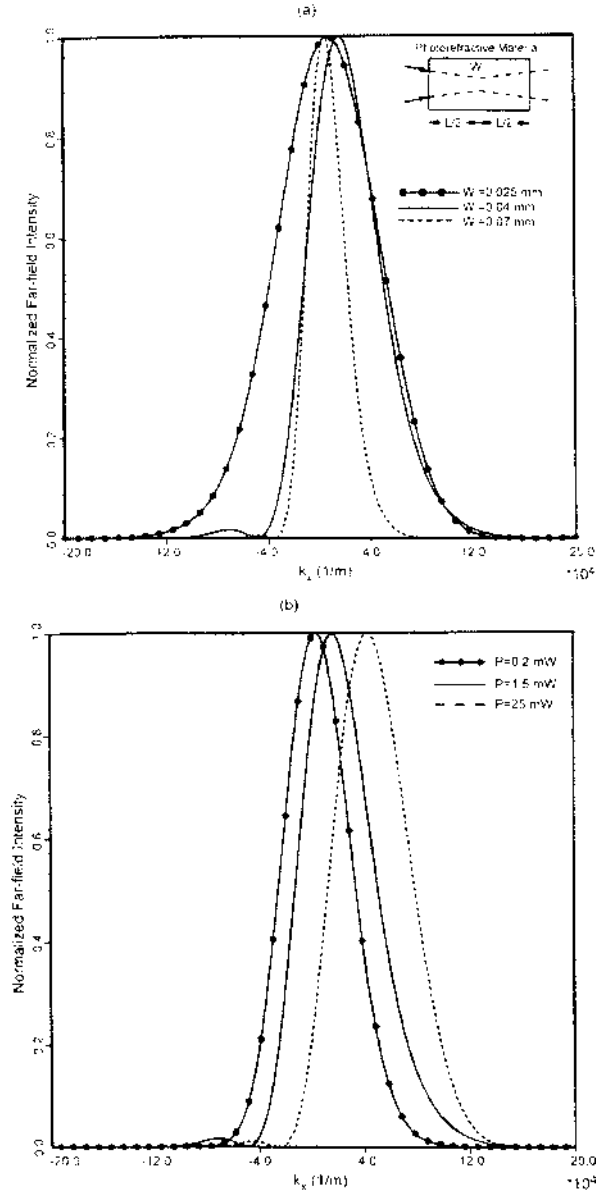
show a monotonic increase in the shift of the far-field main lobe from the  $z$ -axis with increase in  $I_0$  (implying either an increase in power  $P$  or a decrease in width  $W$ ). In Fig. 2a and b,  $k_x$  is the spatial frequency variable corresponding to  $x$  and is related to the far-field coordinate  $x_f$  by  $k_x = k_0 x_f / d$ ,  $d$  being the distance of propagation from the exit face of the PR material to the far field. However, the amount of DBF (defined by the relative amount of power in the sidelobes) varies nonmonotonically with intensity, initially increasing as the intensity  $I_0$  is increased from low levels to attain a maximum, and then decreasing with further increase in intensity.

We will now present the results for the far-field beam profiles using a thick-sample model for the PR material and point out the similarities and differences with the thin-sample model. Numerical simulations for the thick-sample model are performed on the basis of Eq. (3-4) by employing the *split-step beam propagation technique* (see Appendix A). In this simulation, we track both the phase and amplitude modulation of the beam within the crystal due to the combined effects of propagational diffraction (along  $x$  and  $y$ ) and induced refractive index  $\Delta n_e$  (along  $x$ ) arising from the PR effect. Fig. 3a and b shows the normalized far-field intensity patterns with the beam waist  $W$  and power  $P$  as parameters. More precisely,  $W$  denotes the beam waist which would be expected at  $z = L/2$  (i.e., at the center of the sample) in the absence of any PR effect ( $r_{ij} = 0$ ) (see inset in Fig. 3a). The results are qualitatively similar to the thin-sample predictions: DBF is seen to reduce at very low and very high values of  $P$ . Also, DBF reduces at very low and very high values of  $W$ . Quantitatively, for a fixed power  $P$  ( $\cong 1.5$  mW), we can predict the absence of DBF for sufficiently large values for  $W$  ( $\cong 70$   $\mu\text{m}$ ). This prediction is independent of whether a thin- or thick-sample model is used in our simulations. Physically, this makes sense since the thin- and thick-sample models must agree if the diffraction effects in the crystal are sufficiently small. For example, in the limiting case of an incident plane wave, i.e.,  $I = \text{constant}$ , and there is no DBF, since  $\Delta n_e \propto \nabla I = 0$ . On the other hand, the reason for the reduced DBF for a small value of  $W$  in the thick-sample approach is that the width is small only over a very small length (on the order of the Rayleigh range) within the sample. Elsewhere in the sample, the beam width increases as the distance from the waist increases. This increase of the beam width again translates to a reduced PR effect since  $\Delta n_e \propto \nabla I$ .

Unlike  $\text{BaTiO}_3$  where diffusive effects are strong, in  $\text{LiNbO}_3$ , the *photovoltaic effect* is primarily responsible for photorefractivity. Light-induced scattering resulting in DBF has been observed in PR  $\text{LiNbO}_3$  and can be explained on the basis of an induced nonlinear refractive index primarily due to the photovoltaic and thermal effects (Liu et al., 1994). In  $\text{LiNbO}_3$ , the photovoltaic effect is responsible for breaking the circular symmetry of an incident focused extraordinarily polarized Gaussian beam in



**Figure 2** Normalized far field intensity profiles for the thin sample model (a)  $P = 1.5$  mW with  $W$  as a parameter; (b)  $W = 40$  microns with  $P$  as a parameter. The horizontal axis is representative of the distance in the far field. [Banerjee and Misra (1993)].



**Figure 3** Normalized far field intensity profiles for the thick sample model (a)  $P = 1.5$  mW with waist  $W$  as a parameter; (b)  $W = 40$  microns with  $P$  as a parameter. The horizontal axis is representative of the distance in the far field. [Banerjee and Misra (1993)].

the far field, while the thermal effect manifests itself in circularly symmetric far-field patterns (Liu et al., 1994). Over a range of input powers, the photovoltaic effect dominates, resulting in an elongated far-field pattern with the spreading dominant along the  $c$ -axis of the crystal.

The approach we will take to analyze beam propagation in photorefractive LiNbO<sub>3</sub> will be based on the evolution of beam widths of an incident circularly symmetric Gaussian beam focused by a lens onto the material. This will also allow for the prediction of the  $z$ -scan of a sample of the material in order to ascertain the effective nonlinearity, which, as shown below, is linked to the *photovoltaic coefficient* and the acceptor-to-donor concentration in the material. Since we will be considering propagation of focused beams, we will have to assume the LiNbO<sub>3</sub> to be a “thick” sample, i.e., a sample whose thickness is much larger than the Rayleigh range of the focused Gaussian beam. In this case, diffraction effects become important and cannot be neglected. We determine the beam shape as it leaves the nonlinear sample and then calculate the beam profile after it has propagated some distance outside the medium. The information about the effective  $n_2$  is contained in the nature of this profile. In general, the magnitude and sign of the nonlinearity can be determined from the beam profile variation as the sample position is varied about the back focal length of the external lens. The nonlinearity depends on the acceptor-to-donor concentration ratio  $N_A/N_D$ , which in turn determines the far-field diffraction pattern. Conversely, measurements of the far-field pattern can be used to calculate  $N_A/N_D$  and used as a tool for characterizing different LiNbO<sub>3</sub> samples.

Assume therefore an incident Gaussian beam in the form

$$E_e(x, y, z) = a(z) \exp\left(-\frac{x^2}{w_x^2}\right) \exp\left(-\frac{y^2}{w_y^2}\right). \quad (3-8)$$

For an elliptical Gaussian beam, the following relationships hold (compare with Eq. (2-3) in Chap. 4):

$$\Delta q_x = \Delta z + \frac{q_x^2}{f_{\text{ind}_x}}, \quad \Delta q_y = \Delta z + \frac{q_y^2}{f_{\text{ind}_y}}. \quad (3-9)$$

These expressions reduce to the standard expression for the change of the  $q$ -parameter with propagation in a simple nonlinear material, given in Eq. (2-3). Since

$$n = n_e + n_2 |E_e|^2 \approx n_e - 2n_2 a^2(z) \left( \frac{x^2}{w_x^2} + \frac{y^2}{w_y^2} \right) \quad (3-10)$$

where  $n_2$  is the effective nonlinear refractive index coefficient,  $n_e$  is the linear refractive index, and  $E_e$  is the optical field, we can compute the phase change

upon nonlinear propagation through a section  $\Delta z$  of the sample and thereby determine the induced focal length. As expected, these focal lengths are inversely proportional to  $\Delta z$  and can be expressed as:

$$f_{\text{ind}_x} = \frac{n_e w_x^2}{4n_{2_x} a^2(z) \Delta z}, \quad f_{\text{ind}_y} = \frac{n_e w_y^2}{4n_{2_y} a^2(z) \Delta z}. \quad (3-11)$$

Substituting Eq. (3-11) into Eq. (3-9) and taking the limit as  $\Delta z \rightarrow 0$ , we obtain the system of equations

$$\frac{dq_x}{dz} = 1 + \frac{4n_{2_x} a(z) q_x^2}{n_e w_x^2} \quad (3-12)$$

$$\frac{dq_y}{dz} = 1 + \frac{4n_{2_y} a(z) q_y^2}{n_e w_y^2}.$$

Using the well-known relationship  $\frac{1}{q} = \frac{1}{R} + j \frac{\lambda}{n_e \pi w^2}$ , where  $R$  is a radius of Gaussian beam curvature  $\frac{1}{R} = \frac{1}{w} \frac{dw}{dz}$ , and  $\lambda$  is the wavelength in vacuum, we obtain

$$\begin{aligned} \frac{1}{R_x^2} \frac{dR_x}{dz} &= \frac{n_e^2 \pi^2 w_x^4 - \lambda^2 R_x^2}{(n_e \pi w_x^2 R_x)^2} - \frac{4n_{2_x} a^2}{n_e w_x^2}, \\ \frac{1}{R_y^2} \frac{dR_y}{dz} &= \frac{n_e^2 \pi^2 w_y^4 - \lambda^2 R_y^2}{(n_e \pi w_y^2 R_y)^2} - \frac{4n_{2_y} a^2}{n_e w_y^2} \end{aligned} \quad (3-13)$$

$$\begin{aligned} \frac{d^2 w_x}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_x^3} - \frac{4n_{2_x} a^2}{n_e w_x} \\ \frac{d^2 w_y}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_y^3} - \frac{4n_{2_y} a^2}{n_e w_y}. \end{aligned} \quad (3-14)$$

Taking into account the relationship for the beam's power,  $P = (\pi/2\eta) a^2(z) w_x(z) w_y(z)$  where  $\eta$  is the characteristic impedance of the material, which is conserved, we finally have the system of equations describing the Gaussian beam propagation in a thick LiNbO<sub>3</sub> crystal as

$$\begin{aligned} \frac{d^2 w_x}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_x^3} - \frac{8n_{2_x} P \eta}{\pi n_e w_x^2 w_y}, \\ \frac{d^2 w_y}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_y^3} - \frac{8n_{2_y} P \eta}{\pi n_e w_y^2 w_x} \end{aligned} \quad (3-15)$$

Assuming  $n_{2_x} \gg n_{2_y}$  (true for photorefractive lithium niobate), the variation of the widths  $w_x$  and  $w_y$  of an elliptic Gaussian beam propagating

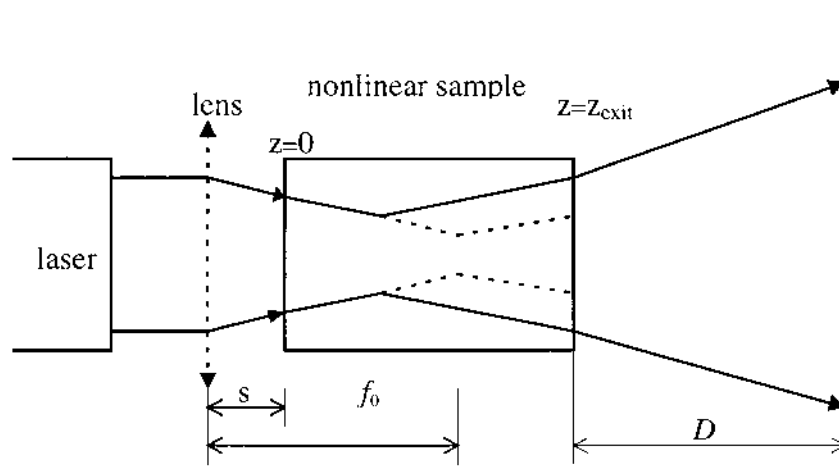
through a thick LiNbO<sub>3</sub> sample as shown in Fig. 4 can be modeled by the coupled differential equations (Banerjee et al., 1998):

$$\begin{aligned}\frac{d^2 w_x}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_x^3} - \frac{8n_2 \eta P}{\pi n_e w_x^2 w_y} \\ \frac{d^2 w_y}{dz^2} &= \frac{\lambda^2}{n_e^2 \pi^2 w_y^3}\end{aligned}\quad (3-16)$$

The case when  $n_{2_x} \neq n_{2_y}$  has been studied in Banerjee and Misra (1993) by employing the  $q$ -transformation approach to find the widths of an elliptic Gaussian beam in a nonlinear medium in the presence of diffraction. Equation (3-16) assumes that the nonlinearity is highly inhomogeneous and only affects the width along the  $x$ -axis (which coincides with the  $c$ -axis of our crystals) due to the large electron mobility along that axis (Song et al., 1993). The effective  $n_2$  can be written as (Liu et al., 1994)

$$n_2 \cong -\frac{1}{2} n_e^3 r_{33} \frac{k \alpha \gamma_R N_A}{\mu e \beta N_D} \quad (3-17)$$

where  $r_{33}$  is the electro-optic coefficient,  $k$  is the photovoltaic constant,  $\alpha$  is the absorption coefficient,  $\gamma_R$  is the recombination constant,  $\mu$  is the mobility,  $e$  is the electron charge, and  $\beta$  is the thermal generation rate. In the above



**Figure 4** Z-scan setup for a thick sample. The thick lines represent the path of the rays, described as the locus of the  $1/e$  points of the Gaussian beam. The thin lines show the ray path in the absence of the medium. Elliptical symmetry of the Gaussian beam is assumed throughout the sample.

equation, we have made the assumption  $\beta \gg sI$  (applicable over the optical power range of interest), where  $s$  is the ionization cross section per quantum of light and  $I$  is the optical intensity.

We will now present analytical and numerical simulation results based on the theory above and compare them with sample experiments using PR LiNbO<sub>3</sub>. If the Gaussian beam incident on the sample is assumed to have planar wave fronts and waist  $w_0$  (approximately at the back focus of the lens), then

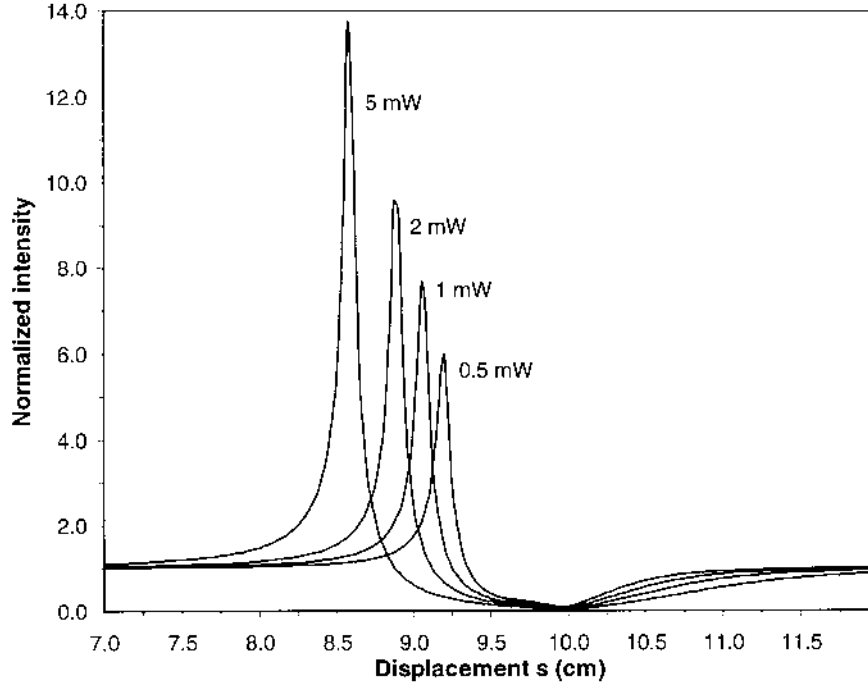
$$w_y^2(z) = w_0^2 \left( 1 + \frac{z^2}{z_{R_y}^2} \right) \quad z_{R_y} = \frac{n_e \pi w_0^2}{\lambda_0}. \quad (3-18)$$

For a sample length  $L$  assumed to be much larger than the Rayleigh ranges  $z_{R_y}$  and  $z_{R_x}$  along  $z$  for the elliptic beam, the evolution of  $w_x$  can be approximated as

$$w_x^2(z) = w_0^2 \left( 1 + \frac{z^2}{z_{R_x}^2} \right) \quad z_{R_x} = \frac{n_e \pi w_0^2}{\lambda_0} \left( 1 + \frac{4n_e n_2 \eta \pi P}{\lambda_0^2} \right). \quad (3-19)$$

It is clear that in the  $x$ -direction, the beam spread is more than that in the linear diffraction-limited case when  $n_2 < 0$  and less when  $n_2 > 0$ . As seen from relation (3-18), the nonlinearity does not affect the beam width along the  $y$ -direction, which leads to elliptic beam cross-section profile at the exit of the crystal and, in general, in the far field.

For more general geometry, where the incident beam does not have a planar wave front, we have solved Eqs. (3-16) numerically. Fig. 5 shows typical  $z$ -scan graphs plotted for four different values of power for the initially circularly symmetric Gaussian beam. In the calculations, we have used the following parameters: crystal width  $L = 10$  mm, lens focal length  $f_0 = 10$  cm,  $\lambda_0 = 514$  nm, initial beam width  $w_0 = 1.0$  mm,  $n_e = 2.20$ ,  $n_2 = -1.4 \times 10^{-12}$  m<sup>2</sup>/V<sup>2</sup>,  $P = 1$  mW, and crystal exit plane to observation plane distance  $D = 1$  m. A simple explanation of the behavior in the limiting case ( $s$  much smaller or larger than  $f_0$ ) seen in Fig. 5 can be given by referring to Fig. 4. When the distance  $s$ , lens-to-sample separation, is much smaller than the lens focal length  $f_0$ , the incident beam is weakly focused and therefore the beam widths lie close to their linear values leading to semilinear diffraction-limited propagation. When  $s$  is much larger than  $f_0$ , the incident beam is weakly diverging and the overall nonlinear effect is small that, in turn, leads to semilinear diffraction-limited propagation. If  $s \sim f_0$ , the incident beam is highly focused and therefore the nonlinear effect

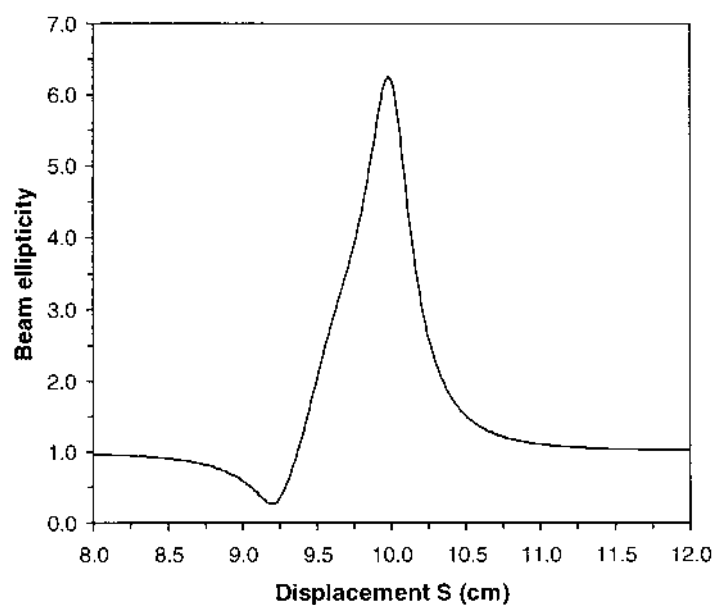


**Figure 5** Typical z-scan graph drawn by solving Eq. (3–16) and propagating the Gaussian beam a distance  $D$  behind the sample.

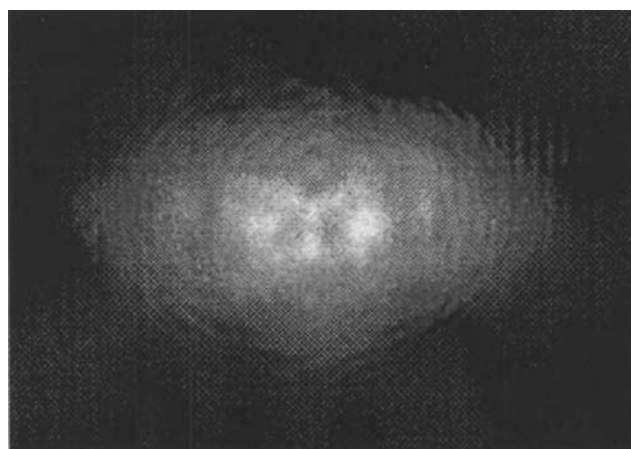
is large. In this region, as  $s$  decreases, the normalized intensity decreases from its linear value, passes through a minimum, and then reaches its maximum before approaching its linear value again. The overall negative slope (between the peak and valley) of the z-scan confirms the net negative nonlinearity of the sample.

Fig. 6 depicts ellipticity  $w_x/w_y$  in the far field vs. displacement  $s$  drawn for the same set of parameters as that used to draw Fig. 5 but for  $P=0.2$  mW. We have done a series of sample experiments and compared results. It turns out that the on-axis intensity measurement of far-field patterns may lead to significant errors due to fine structures in the pattern as seen on Fig. 7 (obtained using a  $\text{LiNbO}_3$  crystal doped with Fe). We have used this crystal for all experimentation to validate our theory, unless otherwise stated. As stated in Sec. 1, possible reasons for this include:

1. Interference patterns stemming from single-beam holography (Kukhtarev et al., 1993).



**Figure 6** Plot of ellipticity as a function of displacement  $s$  for parameters same as in Fig. 4 but for  $P=0.2$  mW.

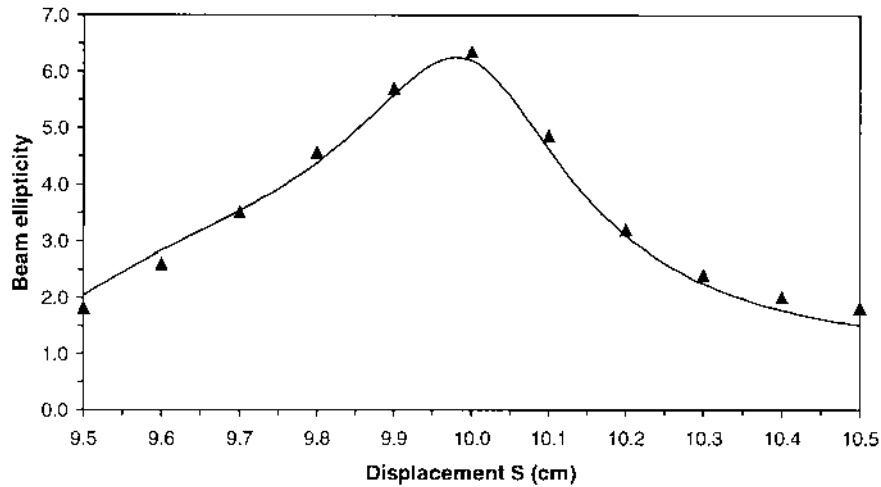


**Figure 7** Typical beam pattern at  $D=0.5$  m for  $P=0.05$  mW,  $f_0=20$  cm, and  $s=19.5$  cm for Fe doped  $\text{LiNbO}_3$  crystal.

2. Interference patterns from optical misalignment.
3. Light diffraction and scattering on crystal defects.
4. Interference patterns from nonparallel crystal edges.

Note that the pattern is approximately symmetric (along  $x$  and  $y$ ). This symmetry arises because the refractive index changes that are due to photovoltaic (and thermal effects) are symmetric and because there is little contribution from diffusion. Experimental results based on the measurement of ellipticity, as shown in Fig. 8, show the same trend as the theoretical predictions superposed on the same figure. The ellipticity was calculated from experimental observations by first determining the extent  $w_x$  and  $w_y$  of the bright or gray region along  $x$  and  $y$ , respectively, from pictures such as Fig. 7 and taking the ratio of the two. Note that Fig. 8 is in fact a blowup of Fig. 6 over the interval 5–10.5 cm. The theoretical graph in Fig. 8 was drawn after examining the experimental results shown in the same figure and choosing that value  $n_2$  for the analytical graph that minimizes the sum of the differences between the experimental points and the corresponding theoretical data.

As a final note, we would like to point out that each time the crystal was displaced along the longitudinal direction for a fresh  $z$ -scan ellipticity measurement, we also made a transverse movement of the crystal in order to



**Figure 8** Experimental (points) and theoretical (line) variation of the beam ellipticity on the observation plane as a function of scan distance. Here,  $P=0.2$  mW,  $D=0.5$  m,  $f_0=10$  cm. Upon comparison,  $n_2=-1.4 \times 10^{-12}$  m<sup>2</sup>/V<sup>2</sup>.

make sure that we were starting out from a virgin location in the crystal for each data point. In other words, we always started out from an initially unexposed region of the crystal and exposed it to the incident illumination until steady state was achieved. Details can be found in Banerjee et al (1998).

#### 4 TWO-WAVE MIXING IN PHOTOREFRACTIVE MATERIALS

As pointed out above, we can approximate the PR response in a diffusion-dominated PR material by way of an induced change in refractive index, according to

$$\Delta n_e = (k_0 r_{\text{eff}} n^3 / 2) (k_B T / e) \nabla I / (\beta / s + I). \quad (4-1)$$

Now we use Eq. (3-3) and write

$$E_e(x, z) = A_1(z) \exp -j(K/2)x + A_{-1}(z) \exp +j(K/2)x \quad (4-2)$$

where we have taken  $A_1$  and  $A_{-1}$  to be real for simplicity. Note that we have assumed the grating vector  $\mathbf{K}$  to be in the  $x$ -direction transverse to the nominal direction of propagation of the waves. Equating coefficients of  $\exp \pm j(K/2)x$ , we obtain, after some straightforward but tedious algebra, the pair of coupled spatial evolution equations for  $A_1(z)$  and  $A_{-1}(z)$  as

$$\begin{aligned} dA_1(z)/dz &= (\gamma/2) A_1 A_{-1}^2 / (A_1^2 + A_{-1}^2) \\ dA_{-1}(z)/dz &= -(\gamma/2) A_1^2 A_{-1} / (A_1^2 + A_{-1}^2) \end{aligned} \quad (4-3)$$

where

$$\gamma = k_0 r_{\text{eff}} n^3 K k_B T / 2e \quad (4-4)$$

and where we have assumed  $(A_1^2 + A_{-1}^2) \gg \eta \beta / s$ , where  $\eta$  is the characteristic impedance of the PR material. It is readily checked upon multiplying the first and second equations in Eq. (4-3) by  $A_1(z)$  and  $A_{-1}(z)$ , respectively, that

$$(A_1^2 + A_{-1}^2) = \text{const.} \quad (4-5)$$

implying conservation of energy. Equation (4-3) can be recast in the form

$$\begin{aligned} dI_1(z)/dz &= \gamma I_1 I_{-1} / (I_1 + I_{-1}) \\ dI_{-1}(z)/dz &= -\gamma I_1 I_{-1} / (I_1 + I_{-1}) \end{aligned} \quad (4-6)$$

where  $I_1$  and  $I_{-1}$  are the intensities of the waves  $A_1$  and  $A_{-1}$ , respectively. The power conservation law translates to  $I_1 + I_{-1} = \text{const.}$

The solutions for the intensities  $I_1$  and  $I_{-1}$  are as follows:

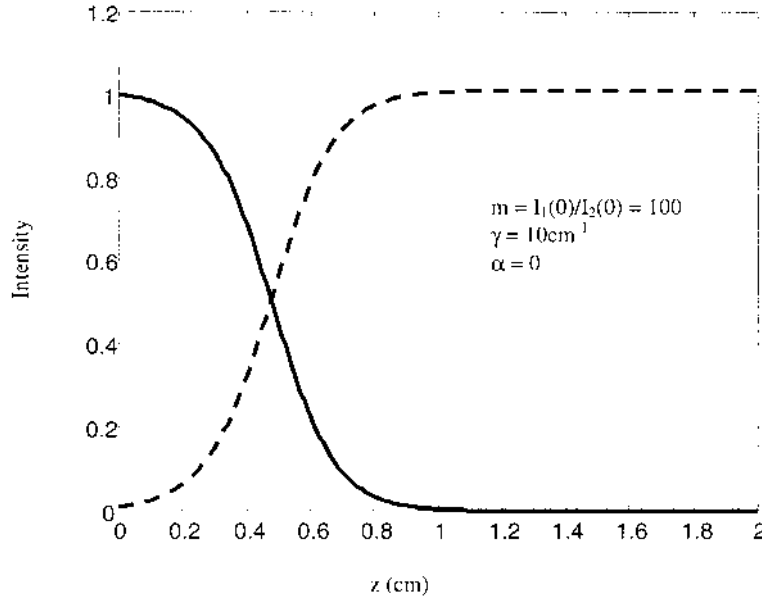
$$\begin{aligned} I_1(z) &= I_1(0) \frac{1 + m^{-1}}{1 + m^{-1}e^{-\gamma z}} \\ I_{-1}(z) &= I_{-1}(0) \frac{1 + m^{-1}}{1 + me^{\gamma z}} \end{aligned} \quad (4-7)$$

where

$$m = I_1(0)/I_{-1}(0). \quad (4-8)$$

The plots of  $I_{1,-1}(z)$  are shown in Fig. 9 as functions of the propagation distance  $z$ . Note that a typical value for  $\gamma L$  is about 10 for a PR material like BaTiO<sub>3</sub>. Also, we remark that in the above analysis, we have left out any effect of attenuation in the material.

Two-wave mixing in PR materials due to induced reflection gratings can be analyzed in a similar way. For simplicity, the analysis is not presented here; the readers are referred to Yeh (1993). Note that mention of this was made in connection with *stimulated Brillouin scattering* in Chap. 7. Suffice to



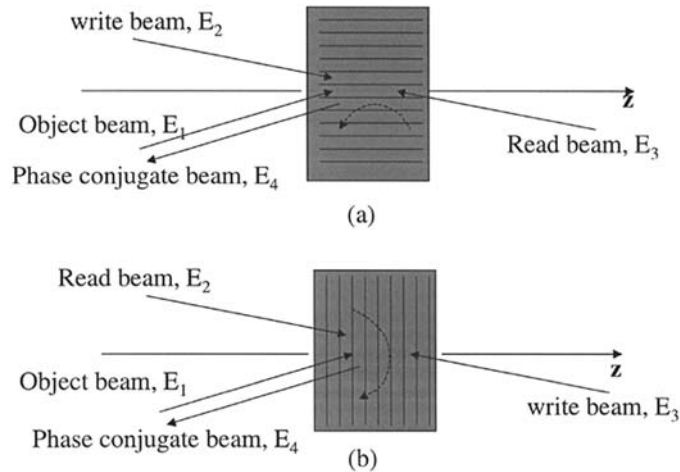
**Figure 9** Energy exchange between two waves during two-wave mixing in a PR material. The attenuation parameter  $\alpha$  is set equal to zero.

state that the starting point is now the Helmholtz equation for the optical phasor instead of Eq. (3-3) for the unidirectionally propagating envelope. The final spatial evolution equations are similar to Eq. (4-6) except that the signs on the right-hand side are the same (e.g., both negative), and the conservation law for power now reads  $I_1 - I_{-1} = \text{const.}$

## 5 FOUR-WAVE MIXING AND PHASE CONJUGATION IN PHOTOREFRACTIVE MATERIALS

The four-wave *phase* conjugation process has a fundamental relationship with conventional *holography*. However, in four-wave mixing, the recording and the readout process are generated at the same time. Thus, we often refer to phase conjugation by four-wave mixing as *real-time holography*. This has been discussed in detail in Chap. 6.

During four-wave mixing in a PR material, there may be two kinds of gratings generated inside the medium at the same time, as shown in Fig. 10. First, *transmission gratings* (Fig. 10a) are caused from interference between *object beam*  $E_1$  and *reference beam*  $E_2$ , which can be called the *writing* beams. The phase conjugate beam will be generated by the diffraction of the read beam  $E_3$  via the transmission grating. We see that the energy of the read beam  $E_3$  transfers through the transmission grating in the PR material



**Figure 10** (a) The transfer of energy from  $E_3$  to  $E_4$  via transmission grating. (b) the backreflection of energy from  $E_2$  to  $E_4$  via reflection grating.

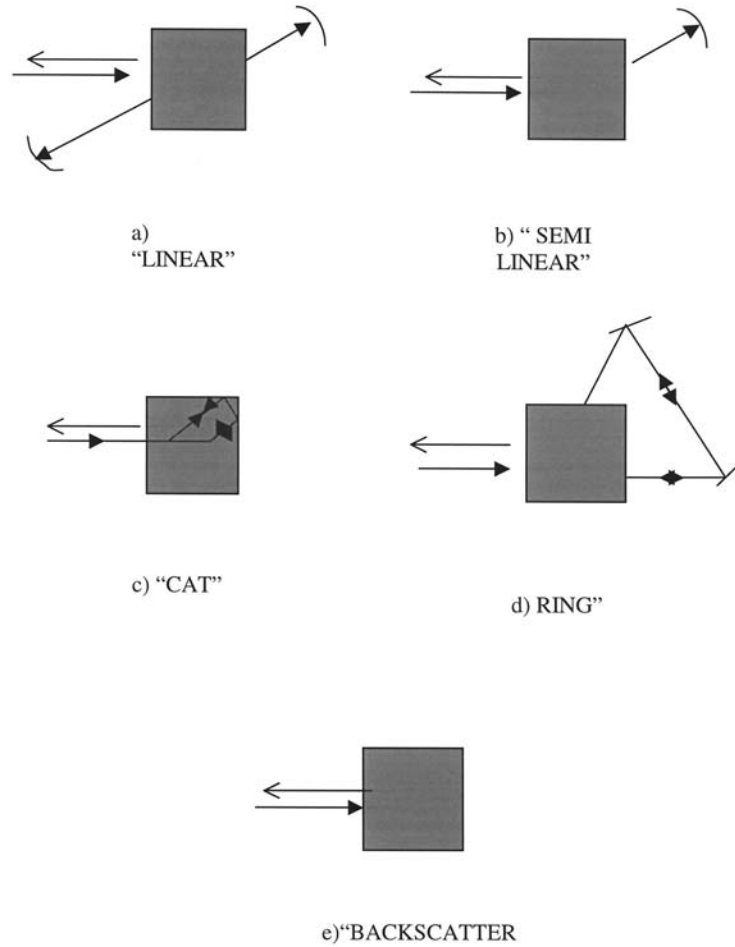
to the phase conjugate beam  $E_4$  in the interaction region. Secondly, *reflection gratings*, as in Fig. 10b, are generated by the interference of the object beam  $E_1$  and beam  $E_3$ , which is now a write beam. The energy of the beam  $E_2$ , which is now a read beam, will be back-reflected via the reflection grating to generate the phase conjugate beam.

Although the reflection grating and transmission gratings may be generated simultaneously, their diffraction efficiencies may vary. If all of the waves have the same frequency, this process is called *degenerate four-wave mixing*. If the frequency of the object beam  $E_1$  is different from the pump beams, the process is *nondegenerate four-wave mixing*. In the reflection grating case (Fig. 10b), if the write beam and object beam counterpropagate exactly, i.e.,  $\mathbf{k}_0 = -\mathbf{k}_w$ , the wave vector  $\mathbf{K} = \mathbf{k}_0 - \mathbf{k}_w = 2\mathbf{k}_0$ , and this kind of grating is referred to as a  $2k$  grating. In this case, the energy also transfers from object beam to phase conjugate beam. Moreover, from Fig. 10b, we see that the object beam and the phase conjugate beam always counterpropagate, so the  $2k$  grating will always be generated.

### 5.1 Photorefractive Self-Pumped Phase Conjugation

An incident beam will produce its own phase conjugate beam in PR phase conjugation. There are two main physical mechanisms that can yield *self-pumped phase conjugation* (SPPC). First, SPPC was generated by resonators that relied on the four-wave mixing method even though all the beams were, at first, from one beam. The first self-pumped phase conjugation created in a PR crystal by White et al. (1982) is an example of a linear resonator. A schematic illustration of the physical arrangement is shown in Fig. 11a. It was made from two mirrors set around a PR crystal as a resonator. The input beam generates two counterpropagating beams, which served as pumping beams. We can remove one of the mirrors as in Fig. 11b after the generation of the phase conjugate beam (Cronin-Golomb et al., 1982a). After that, Feinberg (1982) generated another kind of phase conjugation, but he used the face of the crystal to reflect the beam instead of mirrors as shown in Fig. 11c. This kind of phase conjugation is called the *cat conjugator* because the picture of a cat was used as an image. Later, Cronin-Golomb et al. (1982b) used two mirrors, which make the beam loop back into the crystal, generating phase conjugation as shown in Fig. 11d.

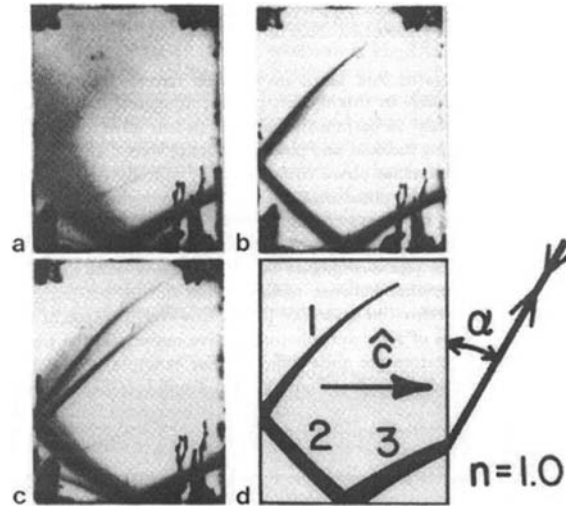
Second, by using the large exponential gain of *stimulated Brillouin scattering* (SBS), which is described in Chap. 8, the phase conjugate beam is generated by large intensities, which can be transferred in two-wave mixing. In a PR crystal such as  $\text{BaTiO}_3$ , the phase conjugate beam due to SBS (sometimes called *stimulated backscattering* because the beam is scattered from the back of the device) arises from two-beam coupling as well (Chang



**Figure 11** Five kinds of self-pumped phase conjugators. a) the "LINEAR" uses two mirrors to form a resonator; b) the "SEMILINEAR" uses only one mirror; c) the "CAT" uses the surface of the crystal to reflect the light instead of mirrors; d) the "RING" uses two mirrors to redirect the beam into the crystal; e) the "BACK-SCATTER" uses stimulated Brillouin scattering (SBS) to reflect the beam.

and Hellwarth, 1985) (see Fig. 11e). This phase conjugation is sometimes known as phase conjugation via a  $2k$  grating. Moreover, there is no reflection at the corner in Chang and Hellwarth's beam traces describing their experiment, as shown in Fig. 12.

Dou et al. (1995) have developed the theory to explain SPPC by four-wave mixing and *stimulated photorefractive backscattering* (SPB).



**Figure 12** Negative photographs of beam traces inside  $\text{BaTiO}_3$  from Chang and Hellwarth (1985). a) Light flooded inside the crystal from beam fanning when the incident beam is first incident in the crystal. b) After a while the fanning beam collapses to a path. Chang et al. assume that this path generates the highest total net gain for backscattered waves. c) For some entrance angles  $\alpha$ , the fanning beam forms multiple beam paths. d) The geometry of a single beam path.

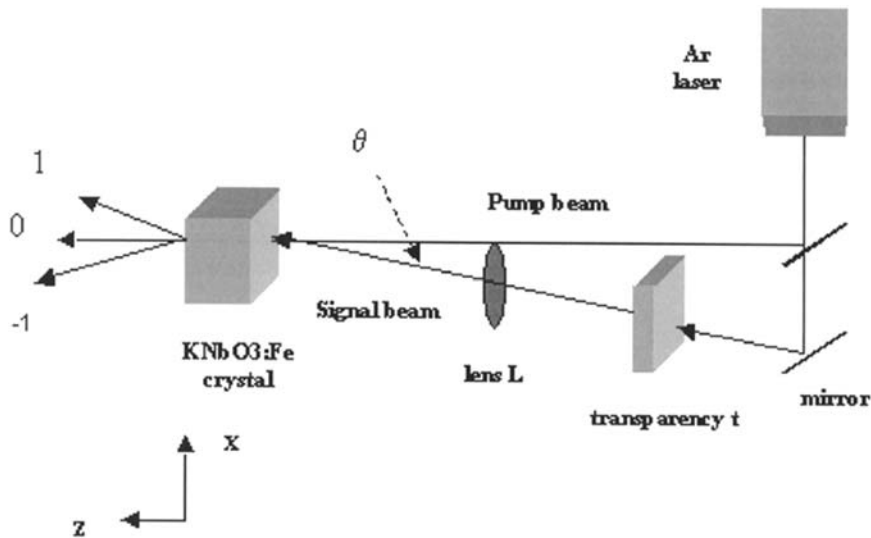
Moreover, they studied the formation mechanisms of SPPC in  $\text{BaTiO}_3\text{:Ce}$  crystals at wavelengths from 570 to 680 nm. In their studies, they used different Ce-doping concentration of 5 and 15 parts in  $10^6$ . They found that the SPPC properties (formation mechanism, efficiency, time response, etc.) of  $\text{BaTiO}_3\text{:Ce}$  can be varied by many factors such as doping concentration, pump wavelength (Lian et al., 1994), crystal-pump geometry, and scattering centers on the bottom face of the crystal. They used the knowledge of increasing the backscattering gain coefficient of a crystal by doping to explain why SPB and SPB-FWM mechanisms form SPPC in doped PR crystals and the total internal reflection (TIR) mechanism forms SPPC in nominally undoped crystals (Dou et al., 1995).

As stated above, there have been three main models proposed to explain the origin of SPPC in  $\text{BaTiO}_3\text{:Ce}$  crystal. First, the model based on the cat phase conjugator shows that SPPC is produced by means of a degenerate four-wave mixing process in two interaction regions inside the crystal. Second, the model based on stimulated PR backscattering shows that SPPC is caused by  $2k$  gratings. The third model comes from a combination of the first and the second models.

## 6 SELF-PHASE CONJUGATION AND EDGE ENHANCEMENT

In this section, we will describe a simple setup that generates the *self-phase conjugate* of an object in the forward direction using a configuration similar to two-wave mixing. Thereafter, we will also show how the self-phase conjugate can also give an *edge-enhanced version* of the image. This scheme for image processing is simple in the sense that no backward propagating beams are involved, and thus it is simple to set it up in the laboratory. The self-phase conjugation discussed here should not be confused with the SPPC described in the earlier section, which occurs in the contrapropagating direction.

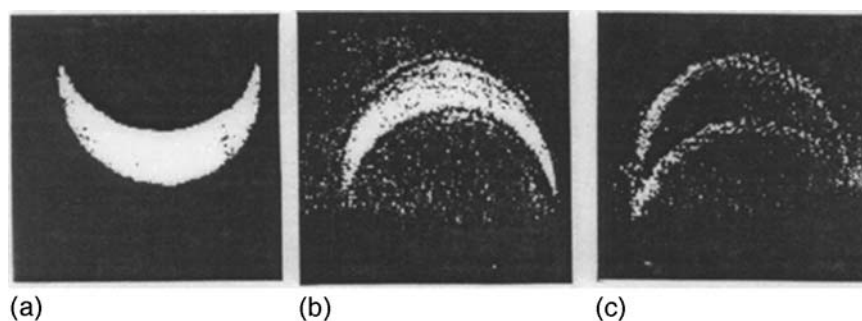
Referring to Fig. 13, we employ the pump beam and the spatial Fourier transform of the object transparency formed by a lens  $L$  of focal length 8.83 cm (also called the signal beam) to write a hologram in the  $\text{KNbO}_3:\text{Fe}$  crystal of dimensions  $5 \times 5 \times 5$  mm (Banerjee et al., 1996). The  $c$ -axis is oriented along the direction of propagation of the pump beam, and the polarizations of all incident beams are horizontal. The pump beam power is 7.5 mW, and the power of the beam illuminating the transparency is 3.0 mW. In traditional phase conjugation, a reading beam counterpropagating to the pump beam (also called the counterpropagating pump beam) gives an output counterpropagating to the signal beam. The phase of this resulting beam is reversed with respect to the signal beam, justifying the



**Figure 13** Experimental setup for observing forward phase conjugation and edge enhancement. Both appear in the  $-1$  order. [Banerjee et al. (1996)].

name *phase conjugate*. In the setup shown in Fig. 13, no counterpropagating pump is used. Instead, the incident pump beam simultaneously reads out the hologram; this generates the conventional 0 and  $+1$  orders, along with a higher ( $-1$ ) order which is the phase conjugate of the object. This is shown in Fig. 14b (monitored at a distance of 1 m behind the crystal), which depicts the phase conjugate of the picture in Fig. 14a, which is our test object. The phase conjugate (as well as the edge-enhanced image, to be discussed later) can be monitored using a laser beam analyzer. For brevity, we have not shown the virtual image of the object (which appears in the  $+1$  direction) and the hexagonal spots produced due to self-diffraction of the pump beam (see Chapter 11). All optical fields behind the crystal are also horizontally polarized. We have taken care in the experiment to ensure that the object is placed exactly at the front focal plane of the lens L. We remark that in practice, if the size of the object is large, one may insert a beam expander before the transparency and employ a confocal two-lens arrangement after the transparency to reduce the effective size of the object before Fourier-transforming with the lens L.

We believe that the reason for the generation of the  $-1$  order in our experiment may be also due to the fact that a *thin hologram* is stored in the  $\text{KNbO}_3$  sample. This is made possible by the fact that the angle between the nominal directions of propagation of the two participating beams is rather large in our case (about  $10^\circ$ ), and furthermore, the finite extent of the Fourier transform of has a small area of overlap with the pump beam. The hologram is stored as a weak phase grating in the material, with the induced change in the refractive index being a function of the intensity of the interference pattern between the pump and the spatial Fourier transform of the object, facilitated through the nonlinear material response of the material. We note that our experiment shows that it is possible to simultaneously record and read a hologram using the same pump beam.



**Figure 14** (a) Picture of the transparency, (b) phase conjugate of the input transparency, (c) edge detection of the input transparency. [Banerjee et al. (1996)].

*Edge enhancement* of the object due to differential *spatial filtering*, as described below, is achieved through a minor change in the experimental setup. The object  $t$  is slightly displaced from the front focal plane of the lens  $L$ . Fig. 14c shows the edge enhancement of our test object, monitored at a distance of about 40 cm behind the crystal. Once again, the optical field comprising the edge is horizontally polarized. The presence of the edge-enhanced output suggests the presence of an induced differential spatial filter in the crystal (see Problem 8).

## 7 ORGANIC PHOTOREFRACTIVE MATERIALS

Recently, there has been a lot of work on a class of PR materials which are inherently different from the ones like barium titanate and lithium niobate discussed above. These are called the *organic photorefractive materials*. There are several reasons for pursuing the development and applications of organic PR materials. One motivation comes from the consideration of a particular *figure-of-merit* that compares the index change possible in different materials (assuming equal densities of trapped charges). This figure-of-merit may be defined as  $Q_e = n^3 r_e / \epsilon_r$  (Yeh, 1993), where  $n$  is the optical index of refraction,  $r_e$  is the effective electro-optic (EO) coefficient, and  $\epsilon_r$  is the d.c. dielectric constant relative to the permittivity of free space  $\epsilon_0$ . Organic PR materials currently in use have a  $Q_e$  of about 3, as compared to about 6 for barium titanate and 10 for lithium niobate. However, there is the potential for much higher figures of merit, and a number often quoted is of the order of 30.

*Doped polymeric PR composites* are particularly attractive because all important functionalities of the PR effect can be optimized simultaneously and almost independently to a degree not possible in the existing inorganic materials. The most important specific advantages of the multicomponent composites approach are as follows.

1. The active wavelength of the material can be tuned to the desired regions of operation by incorporating appropriate sensitizers for charge photogeneration.
2. The charge transfer and trapping characteristics in such materials can also be optimized using dopants with desired ionization energy for charge transfer.
3. Polymeric materials can be processed into different applicable device forms.
4. Optical quality and linear refractive index can be optimized.
5. Ease of synthesizing.
6. Low cost.

An example of a PR polymer is PNP:PVK:ECZ:TNF. The polymer is based on the photoconducting material *poly(N-vinylcarbazole)* (PVK). The electro-optical activity is due to the doping with (*S*)-(-)-*N*-(5-nitro-2-pyridyl)prolinol (PNP) as the optically second-order nonlinear molecules. *N*-ethylcarbazole (ECZ) is a plasticizing agent to lower the *glass transition temperature*  $T_g$ . A small amount of 2-4-7 trinitro-9 fluorenone (TNF) is often added to increase the photosensitivity of the material in the visible region. A typical composition could be PNP:PVK:ECZ:TNF at ratios of 50:33:16:1. The polymer possesses absorption ( $\alpha$ ) of  $25 \text{ cm}^{-1}$  at 633 nm. Organic PR polymers usually have a high gain. For instance, the TBC coefficient ( $\Gamma$ ) for the material is 101 and  $68 \text{ cm}^{-1}$  for s and p-polarized light at 633 nm and applied voltage of  $100 \text{ V}/\mu\text{m}$ , respectively (Matsushita et al., 1997).

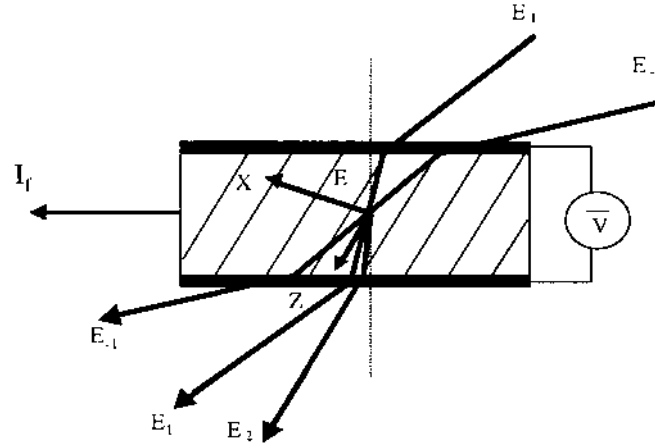
The high gain of PR polymers is advantageous for efficient two-beam coupling and energy exchange; however, strong beam fanning also results from the high-gain coefficient. Furthermore, in a typical two-beam coupling setup, higher-order diffraction is also often observed. The higher-order diffraction is due to the “thin”  $\mathbf{K}$  grating established in the material causing *Raman–Nath-type diffraction* and also due to the presence of  $2\mathbf{K}$  gratings due to the nonlinear dependence of the space charge field on the intensity profile, as well as the nonlinear relationship between the space charge field and the induced refractive index. We remark also that a strong bias is required across the sample for the photorefractive effect.

A typical two-wave coupling setup is shown in Fig. 15. Note that both incident waves are typically at large incident angles so that the intensity grating vector has a component along the direction of the applied electric field. Fig. 15 shows two energy-exchanging waves ( $E_{-1}$ ,  $E_1$ ) and the higher spatial harmonics ( $E_{-2}$ ,  $E_2$ ) in the TWC configuration. In many cases,  $E_{-2}$  exceeds the critical angle and therefore is guided down the film. Also,  $I_f$  represents the fanning beam intensity which is often observed to be guided down the film.

Neglecting diffusion, the time-independent material equations describing charge transport, Coulomb interaction, and photogeneration and recombination of charge carriers for polymer photorefractives (Volodin et al., 1995) can be exactly decoupled in steady state to give

$$\frac{QY}{\left(1 + \frac{dY}{d\xi}\right)} = g \quad (7-1)$$

where  $Y = Y(\xi, \zeta) = E_{st}/E_D$ ,  $\xi = k_D x$ ,  $\zeta = k_D z$ ,  $Q = 1 + I/(I_b + I_d)$ ,  $g = J(\gamma/q\mu g_1 E_D)/(N/N_c - 1)$ ,  $k_D^2 \approx q^2 N_c / \epsilon k_B T$ ,  $E_D^2 \approx k_B T N_c / \epsilon$ . Equation (7-1) is a generalization of the treatment in DelRe et al. (1998). In Eq. (7-1),  $x$  is the direction along the grating vector in the local  $x, z$  frame of the polymer,  $E_{st}$  is



**Figure 15** Two-wave coupling configuration using PR polymer.

the electrostatic field,  $I$  is the intensity profile,  $J$  is the current density,  $I_d$  is the dark irradiance, and  $I_b$  is the background intensity.  $N$  and  $N_c$  are the concentrations of *photosensitive centers* and *compensating charges*, respectively,  $\gamma$  is the *recombination coefficient*,  $\beta$  is the *carrier thermal generation rate*,  $q$  is the electronic charge,  $\epsilon$  is the dielectric constant, and  $\mu$  is the mobility. Using the external boundary condition of the applied voltage  $V$ , and in the limit of negligible diffusion,

$$\begin{aligned} g &\approx (V \cos \theta / E_D L) (1 + Q_0) \sqrt{1 - Q_0^2 m^2 / (1 + Q_0)^2} \\ &\approx (V \cos \theta / E_D L) (1 + Q_0) = Y_0 (1 + Q_0) \end{aligned} \quad (7-2)$$

where  $\theta$  is the angle between the grating vector and the direction of applied voltage,  $Q_0 = I_0 / (I_b + I_d)$  where  $I_0$  is the quiescent intensity,  $L$  is the polymer thickness, and  $m$  ( $< 1$ ) is the initial intensity modulation index. The induced refractive index for the polymer can be thereafter calculated using the relation  $\Delta n = (A/2n) Y^2 E_D^2$  where  $A$  is a constant that includes birefringence and electro-optic effects contributing to orientational enhancement mechanism in nonlinear chromophores in the polymer (Moerner et al., 1994),  $n$  is the refractive index, and  $Y_t$  is the total normalized electrostatic field. For details, the readers are referred to Matsushita et al. (1999).

To analyze the optical propagation in PR polymers, one has to start from the Helmholtz equation for the optical field  $E$ :

$$\nabla^2 E = - \frac{k_0^2}{k_D^2} (n^2 + 2n\Delta n) E, \quad (7-3)$$

and approximate it using the slowly varying envelope approximation (SVEA). Next, the optical field can be expressed as

$$E(\xi, \zeta) = E_1(\xi, \zeta) + E_{-1}(\xi, \zeta) + E_2(\xi, \zeta) + E_{-2}(\xi, \zeta), \quad (7-4)$$

where

$$E_i(\xi, \zeta) = A_i \exp[-j(\kappa_{i\xi}\xi + \kappa_{i\zeta}\zeta)], i = -2, -1, 1, 2 \quad (7-5)$$

$$\kappa_{i\xi, \zeta} = \frac{k_{ix, z}}{k_D}, k_{ix} = \frac{K}{2} = -k_{-ix}, k_{iz} = k_{-iz}, k_{ix}^2 + k_{iz}^2 = n^2 k_0^2. \quad (7-6)$$

Using Eqs. (7-1) (7-2) (7-3) (7-4) (7-5)–(7-6), we can, after lengthy algebra, determine the spatial evolution equations during propagation of the optical fields  $E_i$ ,  $i = -2, -1, 1, 2$ . The effect of phase mismatch must be carefully calculated using the Ewald sphere. Effects of beam fanning can be phenomenologically put in a manner similar to the treatment in Vazquez et al. (1991). The spatial evolution equations can be found in Matsushita et al. (2000). Detailed derivations can be found in Gad (2000). We will show some simulation results below which demonstrate how the two-wave coupling coefficient defined as

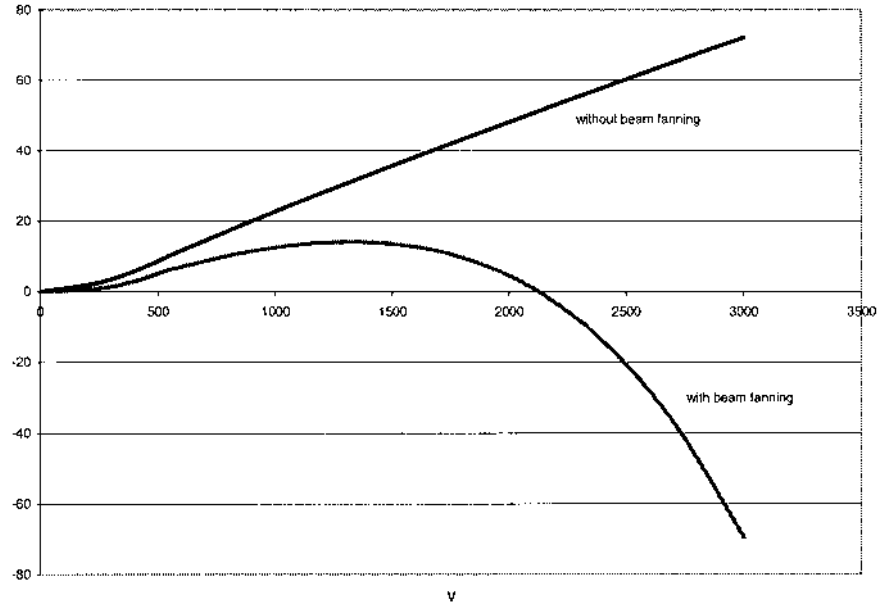
$$\Gamma_1 = \frac{1}{L} \ln \left( \frac{ba}{b+1-a} \right), \quad (7-7)$$

where  $L$  is the length of the interaction region,  $a = \frac{I_1(I_{-1} \neq 0)}{I_1(I_{-1} = 0)}$ , and  $b = \frac{I_{-1}(0)}{I_1(0)}$  varies with applied voltage. As shown in Fig. 16, at low voltages,  $\Gamma_1$  is proportional to the applied voltage. However, with further increase of the applied voltage,  $\Gamma_1$  starts to decrease as more power is lost to beam fanning. When the applied voltage continues to increase even further, the beam gaining power due to energy exchange actually starts to lose power to beam fanning. This is clear from the negative sign of  $\Gamma_1$ . The simulation results agree with the experimental data (Matsushita et al., 2000).

For the purpose of pointing out an interesting potential application, we will write down the evolution equation for a higher (non-Bragg) order. Upon neglecting higher-order gratings and assuming weak higher orders, one can obtain

$$dA_2/d\zeta \propto A_1^2 A_{-1}^* \exp -j\Delta\kappa\zeta, \quad \Delta\kappa = K^2/nk_0k_D. \quad (7-8)$$

It is clear that  $E_2$  contains the phase conjugate of  $E_{-1}$ . We experimentally confirmed this by simulating a point source in the  $-1$  direction (see Fig. 15) using a lens of focal length 10 cm in the path of  $E_{-1}$  and located at a distance of 15 cm before the PR sample. We recovered the real image of the point behind the sample in the direction of the second order. The



**Figure 16** Coupling coefficient  $\Gamma_1$  as a function of applied voltage. [Gad (2000)].

situation is similar to higher-order generation using PR potassium niobate, as discussed earlier. Equation (7-8) also shows that the energy of the +2 order will be approximately proportional to  $\text{sinc}^2 (\Delta\kappa k_D L / \pi \sin \theta)$ , and, to first order, proportional to the square of the applied voltage. Edge enhancement, image correlation, and edge-enhanced image correlation have been demonstrated using PR polymers (Gad, 2000).

## 8 PROBLEMS

1. Linearize the Kukhtarev equations in the steady state through substitutions of the dependent variables and the intensity of the form:  $\psi(x) = \psi_0 + \text{Re}[\psi_1 \exp - jKx]$ . Hence find the spatial frequency dependence of the induced space charge field.
2. Repeat Problem 1 for the case of a moving grating. For this case, use substitution of the form  $\psi(x) = \psi_0 + \text{Re}[\psi_1 \exp - j(\Omega t - Kx)]$ . Once again, find the dependence of the space charge field on the spatial frequency of the grating and the intensity. In the process, define the photorefractive grating decay constant.

3. Investigate contradirectional two-wave mixing through induced reflection gratings in the steady state. Sketch the interacting optical field intensities as a function of interaction length  $z$  in the PR material.
4. In codirectional two-wave coupling in a PR material, show that it is possible to generate higher non-Bragg orders in a so-called “thin” sample. Find the intensities of the first pair of non-Bragg orders.
5. In a material such as lithium niobate where photovoltaic effect is pronounced, the current equation (2.3) has an extra term  $pI$  on the right-hand side, where  $p$  is the photovoltaic constant. Derive the approximate expression [Eq. (3-17)] for the induced refractive index change in this case. If the photovoltaic effect is dominant, is there appreciable energy exchange in two-beam coupling?
6. Determine the variation of the beam ellipticity as a function of the beam power when a focused beam is incident on a sample of lithium niobate. For calculations, use the material parameters used in the text.
7. Design a scheme for optical joint transform correlation using a PR material. The two coupling beams can be referred to as the two object beams which write a hologram in the material. The hologram can be read out by a reading beam. Write down the essential mathematical steps to show that correlation of two objects is possible. Also, show that using a similar scheme, one can obtain self-phase conjugation of an object. (Hint: Poon and Banerjee, 2001.)
8. Edge enhancement is an important image processing technique that can enhance certain higher spatial frequency features of an image. Show why in the experimental setup of Fig. 8 edge enhancement of the object is achieved when the object is displaced from the front focal plane of the lens. (Hint: in diffusion-dominated PR materials, the change in refractive index is proportional to the gradient of the intensity of the interference pattern.)
9. Find the coupled evolution equations for two-wave coupling in a photorefractive polymer. Hence, find the dependence of the coupling constant on the applied voltage.

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# 10

## Nonlinear Optical Properties of Nematic Liquid Crystals

In the last chapter, we discussed photorefractive materials as an example of nonlinear optical materials that can be used for low-power optical processing. Another class of materials that offers potential for low-power nonlinear optical processing is *liquid crystals*, which are more commonly used for everyday applications such as *displays*. In this chapter, we will investigate the nonlinear optical properties of liquid crystals, along with how they interact with incident Gaussian beams.

Research on liquid crystals, which were discovered a century ago, has been very active over the last two decades. Much of the studies are concerned with the characteristic physical properties of liquid crystals. One of these properties, which has recently drawn much attention, is their large optical nonlinearity [Zeldovich et al., 1980; Durbin et al., 1981; Khoo 1982; Tabiryan et al., 1986]. The typical *nematic liquid crystals* have optical nonlinearities  $10^8$  times larger than  $\text{CS}_2$  [Wong and Shen, 1973; Khoo, 1988,]. With this extraordinarily large nonlinearity, several nonlinear optical processes (which are unimaginable in crystals with small nonlinearities), such as wavefront conjugation with gain, self-oscillation, and optical bistability, have been demonstrated using low-power (milliwatt or microwatt) lasers [Khoo et al., 1981].

The study of the nonlinear optical properties of liquid crystals began in the early 1970s. These early studies [Wong and Shen, 1973, 1974] focus on the *isotropic phase* of liquid crystals. It was not until 1979 that the use of the *nematic phase* of liquid crystals for nonlinear optical applications was suggested. Since then, many fundamental and applied studies involving nonlinear optical processes in nematic liquid crystals have been reported [Khoo and Shen 1985; Tabiryan et al., 1986].

## 1 THE LIQUID CRYSTALLINE STATE OF MATTER

The liquid crystalline phase is a state of matter between the solid crystalline phase and the isotropic liquid phase [de Jeu, 1980; Gray and Goodby, 1985; Vertogean and de Jeu 1988]. A liquid crystal can flow like any ordinary liquid, while other properties, such as birefringence, are characteristic of the crystalline phase. This unique combination of properties led to the name *liquid crystal*, which is in itself a contradiction of terms. Other terms used to describe this state of matter are *mesophase* or *mesomorphic phase*.

Liquid crystals can be divided into two classes, the *thermotropic* and the *lyotropic mesophases*. Materials that exhibit mesomorphic behavior within a definite temperature range are termed *thermotropic mesogens*, while those that exhibit mesomorphic behavior in solution are called *lyotropic mesogens*. In the following discussion, only the properties of thermotropic liquid crystals will be discussed in detail because they are most commonly used in the study of nonlinear optical processes.

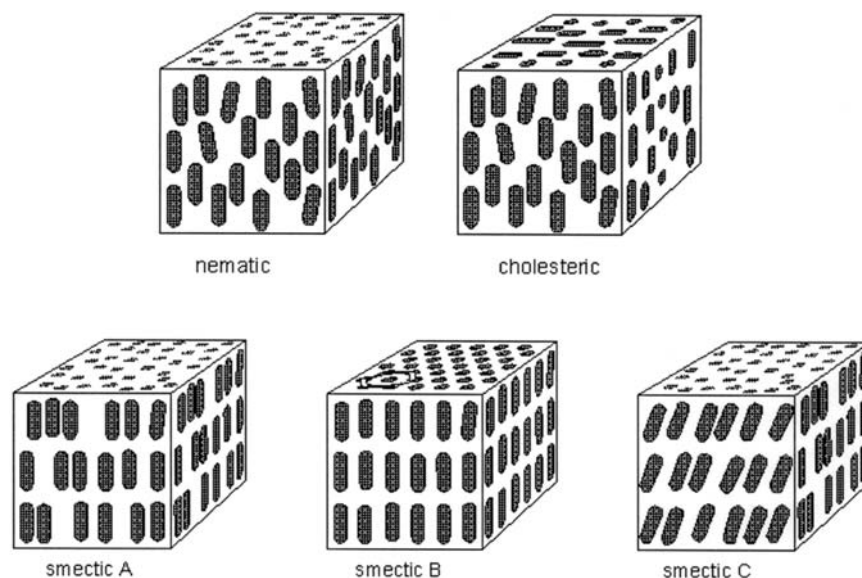
When a thermotropic mesogen is heated, the solid crystalline phase changes to a turbid, strongly birefringent liquid at the melting point. Upon further heating, a second transition point is reached where the highly scattering, turbid liquid becomes an isotropic liquid, which is optically clear. This second transition point is usually referred to as the clearing point and is often characterized by the clearing temperature,  $T_c$ . The melting point and the clearing point define the temperature range for which the mesophase is thermodynamically stable. The phase transitions give rise to interesting optical properties.

## 2 CLASSIFICATION OF LIQUID CRYSTALS

Thermotropic mesogens can exist in three phases: *nematic*, *cholesteric*, and *smectic*, as illustrated in Fig. 1.

The *nematic* phase exhibits long-range molecular orientational ordering but possesses no positional ordering. In this phase, molecules tend to be parallel to some common axis, labeled by a unit vector (or “director”)  $\mathbf{n}$ . An external field, such as electrical field, magnetic field, or optical field, can orient the director when the field strength exceeds the *Fredericksz transition threshold* [Freeder and Zolina, 1933].

A *cholesteric* liquid crystal is thermodynamically equivalent to a nematic liquid crystal except for the chiral-induced twist in the directors, as shown in Fig. 1. If we dissolve in a nematic liquid crystal a molecule that is *chiral* (i.e., different from its mirror image), we find that the structure undergoes a helical distortion. Locally, a cholesteric liquid crystal is very similar to a



**Figure 1** Three phases of thermotropic liquid crystals.

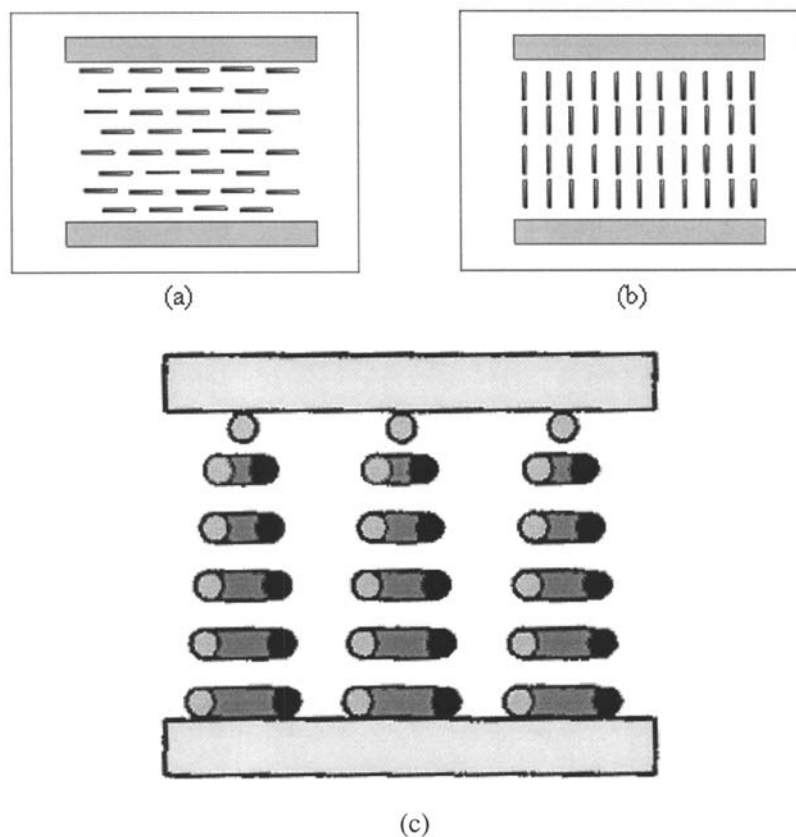
nematic material. Again, the molecular orientation shows a preferred axis labeled by a director,  $\mathbf{n}$ . However,  $\mathbf{n}$  is not constant in space. Instead, it is *helical*. This helical distortion is also found with pure cholesterol esters. For this reason, the helical phase is called cholesteric.

*Smectic* (from the Greek  $\sigma\mu\eta\gamma\mu\alpha$  = soap) is the name coined by Friedel for certain mesophases with mechanical properties reminiscent of soaps. From a structural point of view, all smectics are layered structures with a well-defined interlayer spacing that can be measured by x-ray diffraction. Thus smectics are more ordered than nematics. For a given material, the smectic phases usually occur at temperatures below the nematic domain.

Of the three phases described above, nematic liquid crystals have received the most attention in nonlinear optical research. For this reason, the rest of the discussion in this chapter will focus on the relevant properties of this class of liquid crystals.

### 3 LIQUID CRYSTAL ALIGNMENT

Three basic alignments, *homogeneous* (parallel), *homeotropic* (perpendicular), and *twist* (as illustrated in Fig. 2) have been developed and widely used in nematic liquid crystals [Frederick and Zolina 1933]. Generally, the liquid crystal



**Figure 2** (a) Homogenous alignment. (b) Homeotropic alignment. (c) Twist alignments.

is sandwiched between two optical windows (optical glass for visible wavelengths). On the two inner surfaces of substrates, an electrically conductive but optical transparent metallic film, such as *indium-tin-oxide* (ITO) is coated. The overall alignment of the molecules is determined by the interaction between windows and liquid crystal molecules as well as the interaction between the liquid crystal molecules themselves [Vertoge and de Jeu 1988]. Because the molecules tend to align themselves parallel to one another, we can use microscopic scratches or microgrooves on the substrate surfaces to align them [Uchida and Seki 1992].

In a *homogeneous* sample, the molecules are aligned parallel to the surface of the windows. To produce this alignment, the surface of the windows is carefully rubbed in direction with soft cloth or coated with a thin layer of

SiO<sub>2</sub>. This treatment puts microscopic scratches or microgrooves on the surface of the windows, which cause the boundary layer of liquid crystal molecules to align themselves, parallel to the surface. The intermolecular force causes the surrounding molecules to align themselves with the boundary layer. Placing two substrates in a way that the directions of their microscopic scratches or microgrooves are parallel can align all molecules in the same direction with scratches.

In a *homeotropic* sample, the molecules are aligned with their long axes perpendicular to the surface of the windows. To produce this alignment, the surfaces of the windows are treated with a *surfactant* (hexade-cyltrimethyl-ammonium bromide). This causes the liquid crystal molecules to align themselves normal to the surface of the window. Again, the intermolecular forces cause the surrounding molecules to align themselves in the same direction as the rigidly anchored boundary layer. If the liquid crystal layer is thin (typically  $\leq 200 \mu\text{m}$ ), uniform molecular alignment can be achieved with this technique.

In the *twist* alignment, both substrates are treated similarly to those of parallel alignment except that the back substrate is twisted with an angle.

In practice, homogeneous and homeotropic alignments are commonly used in nonlinear optical studies. Twist alignments are possible but not commonly used. In what follows, we will assume a homogenous aligned nematic liquid crystal sample of typical thickness of about tens of microns.

#### 4 PRINCIPLES OF THE CONTINUUM THEORY

In an ideal, nematic, single crystal, the molecules are (on average) aligned along one common direction  $\mathbf{n}$ . However, in most practical circumstances, this ideal conformation will not be compatible with the constraints that are imposed by the limiting surfaces of the sample (e.g., the walls of a container) and by external fields (magnetic, electric, etc.) acting on the molecules [Vertogeand and de Jeu, 1988]. There will be some deformation of the alignment. This distorted state may be described entirely in terms of a vector field of directors,  $\mathbf{n}(\mathbf{r})$ . The director,  $\mathbf{n}(\mathbf{r})$ , is a unit vector with the direction of the long axis of the molecules at each point. It is assumed that  $\mathbf{n}$  slowly and smoothly varies with  $\mathbf{r}$ .

According the *continuum theory*, the distortion energy per unit volume  $F_{\text{elastic}}$  can be expressed by the so-called *Oseen–Frank equation* [Oseen, 1933; Frank, 1958]:

$$F_{\text{elastic}} = \frac{1}{2} \{ K_{11} (\nabla \cdot \mathbf{n})^2 + K_{22} (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + K_{33} (\mathbf{n} \times \nabla \times \mathbf{n})^2 \} \quad (4-1)$$

where  $K_{11}$ ,  $K_{22}$ , and  $K_{33}$  represent the *splay*, *twist*, and *bend elastic constants*, respectively, and the vector  $\mathbf{n}$  is the unitary liquid crystal director.

To obtain the conditions for equilibrium in the bulk of the liquid crystal, one can postulate that the total distortion energy

$$G = \int F_{\text{elastic}} d\tau \quad (4-2)$$

is a minimum with respect to all variations of the direction  $\mathbf{n}(\mathbf{r})$ .

By using *Lagrange's equation*, one can derive the corresponding *molecular fields* for splay, twist, and bend distortion, respectively, as follows.

$$\mathbf{h}_S = K_{11} \nabla (\nabla \cdot \mathbf{n}) \quad (4-3)$$

$$\mathbf{h}_T = K_{22} [A \nabla \times \mathbf{n} + \nabla \times (A \mathbf{n})] \quad (4-4)$$

$$\mathbf{h}_B = K_{33} [B \nabla \times \mathbf{n} + \nabla \times (\mathbf{n} \times B)] \quad (4-5)$$

where

$$A = \mathbf{n} \cdot (\nabla \times \mathbf{n}) \quad (4-6)$$

$$B = \mathbf{n} \cdot (\nabla \times \mathbf{n}).$$

Consequently, the splay, twist, and bend torques can be found, respectively, by the following equations:

$$\Gamma_S = \mathbf{n} \times \mathbf{h}_S \quad (4-7)$$

$$\Gamma_T = \mathbf{n} \times \mathbf{h}_T \quad (4-8)$$

$$\Gamma_B = \mathbf{n} \times \mathbf{h}_B. \quad (4-9)$$

From the electromagnetic theory, the electrical torque, which acts on the liquid crystal molecule, is

$$\Gamma_E = \mathbf{P} \times \mathbf{E} \quad (4-10)$$

where  $\mathbf{P}$  is electric polarization and  $\mathbf{E}$  is electric field. As we know, from Chap. 1,

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} \quad (4-11)$$

where  $\mathbf{D}$  is the electric displacement.

Then the electrical torque can be expressed by the following form

$$\Gamma_E = \mathbf{D} \times \mathbf{E}. \quad (4-12)$$

Now  $\mathbf{D}$  can be expressed in terms of  $\mathbf{E}$  as:

$$\mathbf{D} = \epsilon_0 \epsilon_{\perp} \mathbf{E} \quad \text{if } \mathbf{E} \text{ is perpendicular to } \mathbf{n}$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\parallel} \mathbf{E} \quad \text{if } \mathbf{E} \text{ is perpendicular to } \mathbf{n}$$

where  $\varepsilon_0$  is the vacuum dielectric constant and  $\varepsilon_{\parallel}$ ,  $\varepsilon_{\perp}$  are the relative dielectric constants parallel and perpendicular to the direction of the molecules, respectively. In general,  $\mathbf{D}$  can be expressed as

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\perp} \mathbf{E} + \varepsilon_0 \Delta \varepsilon (\mathbf{E} \cdot \mathbf{n}) \mathbf{n} \quad (4-13)$$

where  $\Delta \varepsilon \equiv \varepsilon - \varepsilon_{\perp}$ . Thus we can obtain

$$\mathbf{\Gamma}_E = \varepsilon_0 \Delta \varepsilon (\mathbf{E} \cdot \mathbf{n}) (\mathbf{n} \times \mathbf{E}). \quad (4-14)$$

At the equilibrium state, the total torque acting on liquid crystal molecule is equal to 0, i.e.,

$$\mathbf{\Gamma} = \mathbf{\Gamma}_S + \mathbf{\Gamma}_T + \mathbf{\Gamma}_B + \mathbf{\Gamma}_E = 0. \quad (4-15)$$

In general, the director distribution can be found by solving this torque balance equation.

## 5 DIRECTOR DISTRIBUTION OF HOMOGENEOUSLY ALIGNED NEMATIC LIQUID CRYSTAL UNDER AN EXTERNAL ELECTRICAL FIELD

Let us define a Cartesian coordinate system, as shown in Fig. 3. Consider a parallel aligned nematic liquid crystal cell in which the directors are along the  $X$  axis and the two boundary surfaces are at  $z_1 = 0$  and  $z_2 = L$ .  $L$  is the thickness of liquid crystal layer. When the applied voltage (along the  $Z$  axis) exceeds the Freedericksz transition threshold ( $V_0$ ), the directors tilt in the  $X$ - $Z$  plane. The amount of tilt  $\phi(z)$  is a function of the distance from the alignment surface. At the middle of the liquid crystal layer ( $z = L/2$ ),  $\phi(z)$  has maximum value  $\phi_m$  and at the boundaries ( $z = 0$  and  $z = L$ ),  $\phi(z) = 0$ .

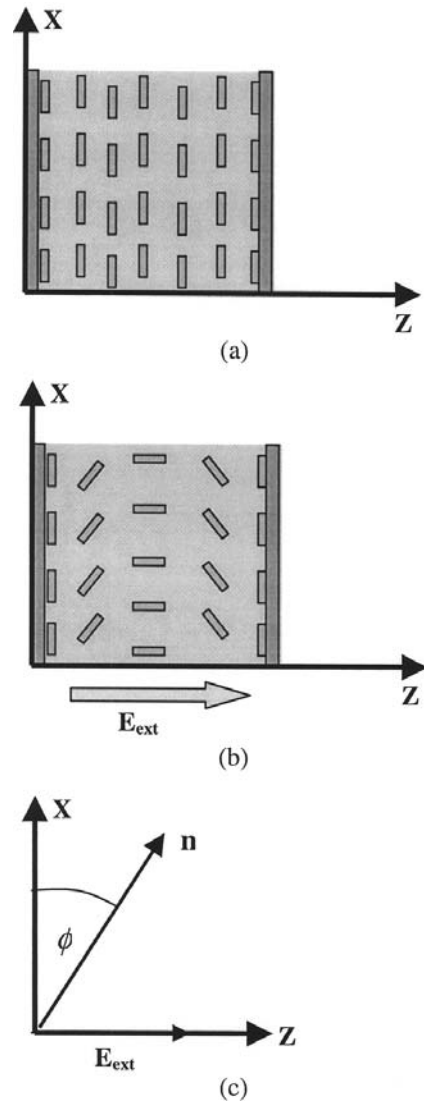
To obtain the director's distribution, we need to find out the splay, twist, bend, and electrical torques in Eqs. (4-7)–(4-9). According to the geometry of Fig. 4, the general form of liquid crystal director,  $\mathbf{n}(z)$ , and external electrical field,  $\mathbf{E} = \mathbf{E}_{\text{ext}}$ , can be expressed by

$$\begin{aligned} \mathbf{n} &= (\cos\phi(z), 0, \sin\phi(z)) \\ \mathbf{E}_{\text{ext}} &= E_{\text{ext}}(0, 0, 1) \end{aligned} \quad (5-1)$$

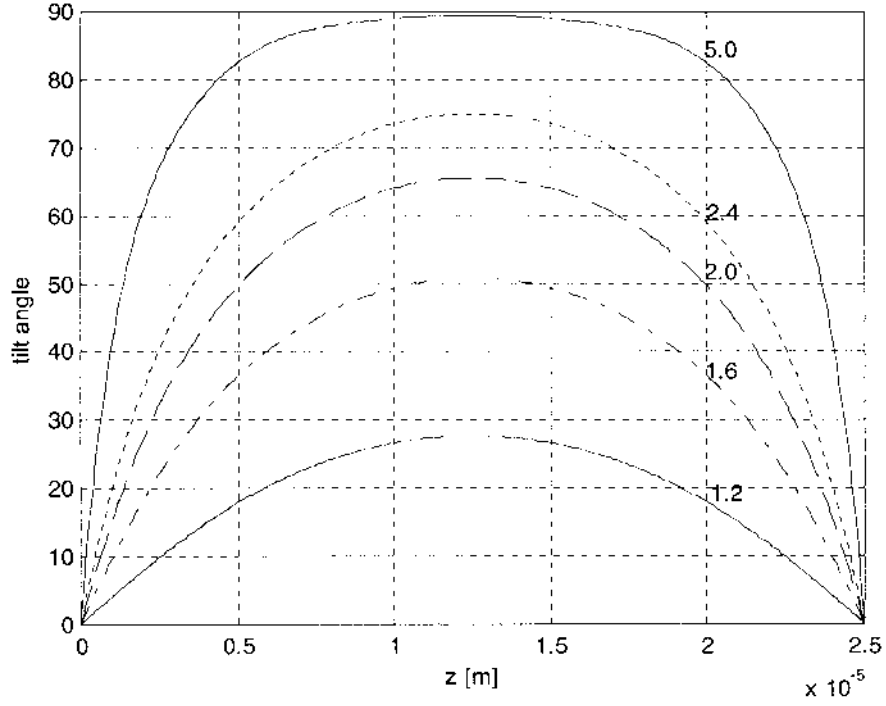
where  $\phi$  is the tilt angle of liquid crystal molecular with respect to the  $X$  axis.

Following Eqs. (4-3)–(4-5), we can find three molecular fields as

$$\mathbf{h}_S = K_{11} \left[ -\sin\phi \left( \frac{d\phi}{dz} \right)^2 + \cos\phi \frac{d^2\phi}{dz^2} \right] \mathbf{a}_z$$



**Figure 3** Liquid crystal cell (a) without  $E_{\text{ext}}$ , (b) with  $E_{\text{ext}}$ , and (c) the coordinate system.



**Figure 4** Director distribution for  $V/V_0 = 1.2, 1.6, 2.0, 2.4, 5.0$ .

$$\mathbf{h}_T = 0$$

$$\begin{aligned} \mathbf{h}_B = & \left[ K_{33} \left( -\cos\phi(-\sin^2\phi + 1) \left( \frac{d\phi}{dz} \right)^2 - \sin\phi \frac{d^2\phi}{dz^2} \right) \right] \mathbf{a}_x \\ & + K_{33} \left[ -\sin^3\phi \left( \frac{d\phi}{dz} \right)^2 \right] \mathbf{a}_z \end{aligned} \quad (5-2)$$

and three elastic torques can be found by Eqs. (4-7)–(4-9) as

$$\Gamma_S = K_{11} \left[ \cos\phi \sin\phi \left( \frac{d\phi}{dz} \right)^2 - \cos^2\phi \frac{d^2\phi}{dz^2} \right] \mathbf{a}_y \quad (5-3)$$

$$\Gamma_T = 0$$

$$\Gamma_B = \left[ K_{33} \left( -\cos\phi \sin\phi \left( \frac{d\phi}{dz} \right)^2 - \sin^2\phi \frac{d^2\phi}{dz^2} \right) \right] \mathbf{a}_y$$

On the other hand, the electrical torque can be found by following Eq. (4-14):

$$\Gamma_E = \Gamma_{\text{ext}} = [-\epsilon_0 \Delta\epsilon E_{\text{ext}}^2 \sin\phi \cos\phi] \mathbf{a}_y \quad (5-4)$$

Substituting Eqs. (5-2) and (5-3) into the torque balance equation, Eq. (4-15), we can obtain

$$\begin{aligned} & (K_{11} \cos^2\phi + K_{33} \sin^2\phi) \left( \frac{d^2\phi}{dz^2} \right) + (K_{33} + K_{11}) \sin\phi \cos\phi \left( \frac{d\phi}{dz} \right)^2 \\ & = -\epsilon_0 \Delta\epsilon E_{\text{ext}}^2 \sin\phi \cos\phi. \end{aligned} \quad (5-5)$$

In principle, the tilt angle  $\phi$  of liquid crystal director can be found by solving Eq. (5-5). However,  $E_{\text{ext}}$  inside liquid crystal cell is yet an unknown function. When the external voltage is just above the threshold voltage  $V_0$ ,  $E_{\text{ext}}(z)$  inside the liquid crystal can be treated as a constant. But for an external voltage much larger than threshold voltage  $V_0$  we do not have any knowledge about  $E_{\text{ext}}(z)$ . The only thing we know about the external electrical field is the applied voltage. Deuling has derived another differential equation in terms of the external applied voltage instead of using electrical field [Deuling 1972, 1978]. This is discussed below.

Starting with Eq. (4-1), the three energy density terms can be found through the following calculation.

$$\begin{aligned} \nabla \cdot \mathbf{n} &= \cos\phi \frac{d\phi}{dz} \\ \mathbf{n} \cdot (\nabla \times \mathbf{n}) &= 0 \\ \mathbf{n} \times (\nabla \times \mathbf{n}) &= \left[ \sin^2\phi \frac{d\phi}{dz}, 0, -\sin\phi \cos\phi \frac{d\phi}{dz} \right]. \end{aligned} \quad (5-6)$$

Then the elastic free energy per unit volume is obtained as

$$F_{\text{elastic}} = \frac{1}{2} (K_{11} \cos^2\phi + K_{33} \sin^2\phi) \left( \frac{d\phi}{dz} \right)^2. \quad (5-7)$$

As we know, the electrical energy density is

$$F_{\text{electric}} = -\frac{1}{2} \mathbf{D} \cdot \mathbf{E} \quad (5-8)$$

where  $\mathbf{D}$  is the dielectric displacement vector and  $\mathbf{E}$  is the electric field.

Consequently, the total free energy is obtained as

$$F_{\text{total}} = F_{\text{elastic}} + F_{\text{electric}} = \frac{1}{2} (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 - \frac{1}{2} \mathbf{D} \cdot \mathbf{E}. \quad (5-9)$$

For the one-dimensional case, the free energy per unit area is

$$G = \frac{1}{2} \int_0^d (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 dz - \frac{1}{2} \int_0^d \mathbf{D} \cdot \mathbf{E} dz. \quad (5-10)$$

For  $\mathbf{D}$  and  $\mathbf{E}$ , we further have the equations (see Chap. 1)

$$\nabla \cdot \mathbf{D} = 0 \quad (5-11a)$$

$$\nabla \cdot \mathbf{E} = 0. \quad (5-11b)$$

The symmetry of the problem requires that all quantities be a function of  $z$  only. From  $\nabla \cdot \mathbf{D} = 0$ , we then immediately see that the  $z$  component  $D_z$  of  $\mathbf{D}$  is a constant. Consequently, we can obtain

$$\frac{1}{2} \int_0^d \mathbf{D} \cdot \mathbf{E} dz = \frac{1}{2} D_z \int_0^d E dz = \frac{1}{2} D_z V \quad (5-12)$$

where  $V$  is the external voltage.  $D_z$  is independent of the coordinate  $z$  but depends on  $\phi$ , that is to say  $D_z$  is a function of  $\phi$ . Because  $\nabla \times \mathbf{E} = 0$ ,  $\mathbf{E}$  can be expressed as the gradient of some potential. By symmetry, this potential can only depend on  $z$ , that is to say only the  $z$  component  $E_z$  of  $\mathbf{E}$  is different from zero. We split  $D_z$  into components parallel ( $D_{\parallel}$ ) and perpendicular ( $D_{\perp}$ ) to the direction of molecules:

$$D_{\parallel} = \epsilon_0 \epsilon_{\parallel} E_z \sin \phi \quad (5-13)$$

$$D_{\perp} = \epsilon_0 \epsilon_{\perp} E_z \cos \phi.$$

For  $D_z$  we have

$$D_z = D_{\parallel} \sin \phi + D_{\perp} \cos \phi. \quad (5-14)$$

Substituting Eq. (5-13) into (5-14), we obtain

$$E_z = \frac{D_z}{\epsilon_0 (\epsilon_{\parallel} \sin^2 \phi + \epsilon_{\perp} \cos^2 \phi)}. \quad (5-15)$$

Using Eq. (5-15) and the definition of voltage

$$V = \int_0^d E dz = \int_0^d E_z dz \quad (5-16)$$

we obtain

$$V = \frac{D_z}{\varepsilon_0} \int_0^d (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} dz. \quad (5-17)$$

Substituting Eq. (5-17) into (5-10), we obtain

$$G = \frac{1}{2} \int_0^d (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 dz \quad (5-18)$$

$$- \frac{D_z^2}{2\varepsilon_0} \int_0^d (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} dz.$$

Taking the variation of this expression with respect to  $\phi(z)$ , and setting it to zero, i.e.

$$\partial G = \frac{\partial G}{\partial \phi} d\phi = 0 \quad (5-19)$$

we obtain

$$\frac{\partial G}{\partial \phi} = \frac{\partial}{\partial \phi} \left\{ \frac{1}{2} \int_0^d \left[ (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 - \frac{D_z^2}{\varepsilon_0} (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} \right] dz \right\} = 0 \quad (5-20)$$

so that

$$\frac{\partial}{\partial \phi} \left[ (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 - \frac{D_z^2}{\varepsilon_0} (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} \right] = 0. \quad (5-21)$$

Integrating Eq. (5-21) once yields

$$\int \frac{\partial}{\partial \phi} \left[ (K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 - \frac{D_z^2}{\varepsilon_0} (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} \right] d\phi = C \quad (5-22)$$

or

$$(K_{11} \cos^2 \phi + K_{33} \sin^2 \phi) \left( \frac{d\phi}{dz} \right)^2 - \frac{D_z^2}{\varepsilon_0} (\varepsilon_{\parallel} \sin^2 \phi + \varepsilon_{\perp} \cos^2 \phi)^{-1} = C \quad (5-23)$$

where  $C$  is the integration constant to be determined. Applying the condition  $d\phi/dz = 0$  and  $\phi(L/2) = \phi_m$  for  $z = L/2$  to Eq. (5-23), we have

$$- \frac{D_z^2}{\varepsilon_0} (\varepsilon_{\parallel} \sin^2 \phi_m + \varepsilon_{\perp} \cos^2 \phi_m)^{-1} = C. \quad (5-24)$$

Substituting Eq. (5-24) into Eq. (5-23) and after rearranging, we have

$$\left(\frac{d\phi}{dz}\right)^2 = \frac{1}{(K_{11}\cos^2\phi K\sin^2\phi)} \times \left[ \frac{D_z^2/\varepsilon_0}{(\varepsilon_{\parallel}\sin^2\phi + \varepsilon_{\perp}\cos^2\phi)} - \frac{D_z^2/\varepsilon_0}{(\varepsilon_{\parallel}\sin^2\phi_m + \varepsilon_{\perp}\cos^2\phi_m)} \right]. \quad (5-25)$$

Introducing constants

$$\begin{aligned} \rho &= \varepsilon_{\parallel}/\varepsilon_{\perp} - 1 \\ \kappa &= K_{33}/K_{11} - 1 \end{aligned} \quad (5-26)$$

Equation (5-25) can be rewritten as

$$\frac{d\phi}{dz} = \frac{D}{\sqrt{\varepsilon_0\varepsilon K_{11}/\rho}} \left[ \frac{\sin^2\phi_m - \sin^2\phi}{(1 + \kappa\sin^2\phi)(1 + \rho\sin^2\phi_m)(1 + \rho\sin^2\phi)} \right]^{1/2}. \quad (5-27)$$

Consequently, we can obtain

$$z \frac{\sqrt{\varepsilon_0\varepsilon K_{11}}}{D_z\sqrt{\rho}} \sqrt{1 + \rho\sin^2\phi_m} \int_0^{\phi} \sqrt{\frac{(1 + \kappa\sin^2\phi)(1 + \rho\sin^2\phi)}{\sin^2\phi_m - \sin^2\phi}} d\phi. \quad (5-28)$$

Using the condition  $\phi(L/2) = \phi_m$  again, we have

$$D_z \frac{2\sqrt{\varepsilon_0\varepsilon K_{11}}}{\sqrt{\rho}} \sqrt{1 + \rho\sin^2\phi_m} \int_0^{\phi_m} \sqrt{\frac{(1 + \kappa\sin^2\phi)(1 + \rho\sin^2\phi)}{\sin^2\phi_m - \sin^2\phi}} d\phi. \quad (5-29)$$

Substituting Eq. (5-28) into Eq. (5-27), we obtain

$$\begin{aligned} \varepsilon_0 V &= \frac{2\sqrt{\varepsilon_0\varepsilon K_{11}}}{L\sqrt{\rho}} \sqrt{1 + \rho\sin^2\phi_m} \\ &\times \int_0^d (\varepsilon_{\parallel}\sin^2\phi + \varepsilon_{\perp}\cos^2\phi)^{-1} dz \int_0^{\phi_m} \sqrt{\frac{(1 + \kappa\sin^2\phi)(1 + \rho\sin^2\phi)}{\sin^2\phi_m - \sin^2\phi}} d\phi. \end{aligned} \quad (5-30)$$

After some algebra, Eq. (5-30) can be rewritten as

$$V = 2\sqrt{\frac{K_{11}}{(\varepsilon_{\parallel} - \varepsilon_{\perp})\varepsilon_0}} \sqrt{1 + \rho\sin^2\phi_m} \int_0^{\phi_m} \sqrt{\frac{1 + \kappa\sin^2\phi}{(1 + \rho\sin^2\phi)(\sin^2\phi_m - \sin^2\phi)}} d\phi. \quad (5-31)$$

Upon defining the threshold voltage  $V_0$  as

$$V_o = \pi \sqrt{\frac{K_{11}}{\epsilon_0 \Delta \epsilon}} \quad (5-32)$$

Equation (5-31) can be expressed as

$$\frac{V}{V_o} = \frac{2}{\pi} \sqrt{1 + \rho \sin^2 \phi_m} \int_0^{\phi_m} \sqrt{\frac{1 + \kappa \sin^2 \phi}{(1 + \rho \sin^2 \phi)(\sin^2 \phi_m - \sin^2 \phi)}} d\phi. \quad (5-33)$$

Finally, upon dividing Eq. (5-28) by (5-29), we obtain

$$\frac{2z}{L} = \int_0^{\phi_m} \sqrt{\frac{(1 + \kappa \sin^2 \phi)(1 + \rho \sin^2 \phi)}{(\sin^2 \phi_m - \sin^2 \phi)}} d\phi = \int_0^{\phi} \sqrt{\frac{(1 + \kappa \sin^2 \phi)(1 + \rho \sin^2 \phi)}{(\sin^2 \phi_m - \sin^2 \phi)}} d\phi. \quad (5-34)$$

Equations (5-33) and (5-34) can be used to solve  $\phi(z)$  for every  $V$ . For any given value of  $V/V_0$ , we can find the corresponding  $\phi_m$  from Eq. (5-33). For any given value of  $z$ , we can then solve  $\phi$  from Eq. (5-34) with the value  $\phi_m$  from Eq. (5-33).

As noted above, we can evaluate the complete tilt angle profile using Eqs. (5-33) and (5-34). However, a mathematical difficulty appears in the integrands of these equations because of the factor  $(\sin^2 \phi_m - \sin^2 \phi)$  in the denominator. As  $\phi$  approaches  $\phi_m$ , this factor creates a singularity in the integrand. To avoid this problem, Deuling (1972) proposed straightforward changes of variables by substituting

$$\begin{aligned} \sin \phi &= \sin \phi_m \sin \Psi \\ \chi &= \sin^2 \phi_m. \end{aligned} \quad (5-35)$$

After some algebra, Eqs. (5-33) and (5-34) can be expressed as

$$\frac{V}{V_o} = \frac{2}{\pi} \sqrt{1 + \rho \chi} \int_0^{\pi/2} \sqrt{\frac{1 + \kappa \chi \sin^2 \Psi}{(1 + \rho \chi \sin^2 \Psi)(1 - \chi \sin^2 \Psi)}} d\Psi \quad (5-36)$$

and

$$\begin{aligned} \frac{2z}{L} &= \int_0^{\pi/2} \sqrt{\frac{(1 + \kappa \chi \sin^2 \Psi)(1 + \rho \chi \sin^2 \Psi)}{1 - \chi \sin^2 \Psi}} d\Psi \\ &= \int_0^{\sin^{-1}(\sin \phi / \sqrt{\chi})} \sqrt{\frac{(1 + \kappa \chi \sin^2 \Psi)(1 + \rho \chi \sin^2 \Psi)}{1 - \chi \sin^2 \Psi}} d\Psi. \end{aligned} \quad (5-37)$$

For any given value of  $V/V_0$ , we can find the corresponding  $\chi$  from Eq. (5-36). For any given value of  $z$  from Eq. (5-37), we can then solve  $\Psi$  with the value  $\chi$ . Once  $\Psi$  is known, the tilt angle  $\phi$  can be found by Eq. (5-37). Equations (5-36) and (5-37) may be conveniently used for any values of  $z$  and  $\Psi$  provided  $\chi$  is not very close to 1. Clearly, as  $\chi$  approaches 1 (that is  $\phi_m$  approaches  $\pi/2$ ), once again infinities appear in the integrands because of the  $(1 - \chi \sin^2 \Psi)$  factor in the denominator. This happens when  $V/V_0 > 3$ . To avoid this problem, we follow Welford (1987) and introduce another two parameters

$$Y = \tan^2 \phi_m = \frac{\chi}{1-\chi} \quad (5-38)$$

$$W = \tan^2 \Psi$$

and rewrite Eqs. (5-36) and (5-37) as:

$$\frac{V}{V_{th}} = \frac{1}{\pi} \sqrt{1 + Y(1 + \rho)} \quad (5-39)$$

$$\int_0^\infty \sqrt{\frac{(1+Y)(1+W) + \kappa YW}{[(1+Y)(1+W) + \rho YW](1+Y+W)(1+W)W}} dW$$

and

$$\begin{aligned} \frac{2z}{L} \int_0^\infty \sqrt{\frac{[(1+Y)(1+W) + \kappa YW][(1+Y)(1+W) + \rho YW]}{(1+Y+W)(1+W)W}} \frac{dW}{1+W} \\ = \int_0^{W_t} \sqrt{\frac{[(1+Y)(1+W) + \kappa YW][(1+Y)(1+W) + \rho YW]}{(1+Y+W)(1+W)W}} \frac{dW}{1+W} \end{aligned} \quad (5-40)$$

where  $W_t$ , the upper limit in the last integral, is given by

$$W_t = \frac{\tan^2 \phi (1 + Y)}{Y - \tan^2 \phi}. \quad (5-41)$$

In principle, upon knowing the liquid crystal material constants, one can evaluate the  $Y$  in Eq. (5-39) for a given applied voltage. Once  $Y$  is obtained, the complete profile of  $\phi(z)$  can be calculated by solving  $W_t$  in Eq. (5-40). Once  $W_t$  is known, the tilt angle  $\phi$  can be found by Eq. (5-41).

Fig. 4 shows the calculated results of the tilt angle,  $\phi(z)$ , of liquid crystal director for external voltage  $V/V_0 = 1.01, 1.2, 1.6, 2.0, 2.4$ , and 5 with  $\rho =$

2.278 and  $\kappa = 0.534$ , where  $V_0$  is the threshold voltage. It is clear that the maximum tilt angle increases as the external voltage increases.

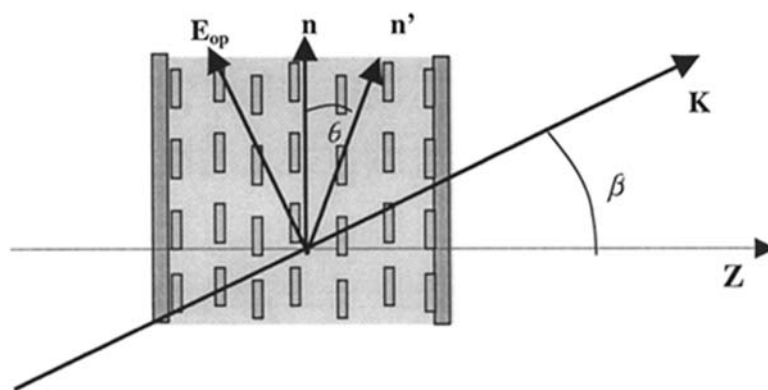
## 6 NONLINEAR OPTICAL PROPERTIES FROM OPTICALLY INDUCED MOLECULAR REORIENTATION

The high nonlinearities of liquid crystals originate from the optically induced molecular reorientation and the laser-induced thermal effect. In this chapter, we will only concentrate on *optically induced molecular reorientation* assuming that thermal effects are small.

Similar to the case of an externally applied electrical field, the reorientation of the director by an applied optical field results from the system's tendency to assume a configuration with minimum *Gibbs free energy*. By applying the continuum theory, we can find the reorientation of the director.

Consider the geometry of interaction as depicted in Fig. 5, where a linearly polarized laser is incident on a homogeneously aligned nematic liquid crystal. We assume no external electrical bias for now. Extension to the case where both external bias and optical field are present is discussed later. The propagation vector  $\mathbf{k}$  of the laser makes an angle  $(\pi/2 - \beta - \theta)$  with the perturbed director axis, where  $\theta$  is the reorientation angle and  $\beta$  is the angle between  $\mathbf{k}$  and the  $Z$  axis. According to this geometry, the perturbed director  $\mathbf{n}'$  and the optical field  $\mathbf{E}_{op}$  can be expressed in the following form:

$$\begin{aligned}\mathbf{n}' &= (\cos\theta(z), 0, -\sin\theta(z)) \\ E_{op} &= E_{op}(\sin\beta, 0, -\cos\beta).\end{aligned}\tag{6-1}$$



**Figure 5** Interaction of a linearly polarized (extraordinary) laser with a homogeneously aligned nematic liquid crystal film.

The three molecular fields are obtained by Eqs. (4-3)–(4-5) as

$$\begin{aligned}
 \mathbf{h}_S &= K_{11} \left[ \sin\theta \left( \frac{d\theta}{dz} \right)^2 - \frac{d^2\theta}{dz^2} \right] \mathbf{a}_z \\
 \mathbf{h}_T &= 0 \\
 \mathbf{h}_B &= K_{33} \left[ (-\sin^2\theta + 1) \cos\theta \left( \frac{d\theta}{dz} \right)^2 - \sin\theta \frac{d^2\theta}{dz^2} \right] \mathbf{a}_x \\
 &\quad + K_{33} \left[ -\sin^3\theta \left( \frac{d\theta}{dz} \right)^2 \right] \mathbf{a}_z.
 \end{aligned} \tag{6-2}$$

Therefore the three elastic torques, i.e., splay, twist, and bend, are obtained by Eqs. (4-7)–(4-9) as

$$\begin{aligned}
 \Gamma_S &= K_{11} \left[ -\cos\theta \sin\theta \left( \frac{d\theta}{dz} \right)^2 + \cos^2\theta \frac{d^2\theta}{dz^2} \right] \mathbf{a}_y \\
 \Gamma_T &= 0 \\
 \Gamma_B &= K_{33} \left[ \cos\theta \sin\theta \left( \frac{d\theta}{dz} \right)^2 + \sin^2\theta \frac{d^2\theta}{dz^2} \right] \mathbf{a}_y
 \end{aligned} \tag{6-3}$$

Similar to the electrical torque, the optical torque can be derived by Eq. (4-14) as

$$\Gamma_{op} = [\varepsilon_0 \Delta\varepsilon \langle E_{op}^2 \rangle \sin(2\beta + 2\theta)] \mathbf{a}_y. \tag{6-4}$$

Substituting Eqs. (6-3) and (6-4) into Eq. (4-15), we can obtain the following differential equation:

$$\begin{aligned}
 &(K_{11} \cos^2\theta + K_{33} \sin^2\theta) \left( \frac{d^2\theta}{dz^2} \right) + (K_{33} - K_{11}) \sin\theta \cos\theta \left( \frac{d^2\theta}{dz^2} \right) \\
 &= -\varepsilon_0 \Delta\varepsilon \langle E_{op}^2 \rangle \sin(2\beta + 2\theta).
 \end{aligned} \tag{6-5}$$

If we assume  $K_{11} = K_{33} = K$ , then the differential equation can be simplified to the following form:

$$K \frac{d^2\theta}{dz^2} + \varepsilon_0 \Delta\varepsilon \langle E_{op}^2 \rangle \sin(2\beta + 2\theta) = 0. \tag{6-6}$$

In the small  $\theta$  approximation, i.e.

$$\sin\theta \equiv \theta$$

$$\cos\theta \equiv 1$$

Equation (6-6) can be written as

$$2\xi^2 \frac{d^2\theta}{dz^2} + (2\cos 2\beta) \cdot \theta + \sin 2\beta = 0 \quad (6-7)$$

where

$$\xi^2 = \frac{K}{\varepsilon_0 \Delta \varepsilon \langle E_{\text{op}}^2 \rangle}. \quad (6-8)$$

Using the so-called *hard-boundary conditions*, i.e., the director axis is not perturbed at the boundary ( $\theta = 0$  at  $z = 0$  and at  $z = L$ ), we can solve  $\theta$  from Eq. (6-7). Upon defining

$$\varphi = (2\cos 2\beta) \cdot \theta + \sin 2\beta, \quad (6-9)$$

the boundary conditions become

$$\varphi(0) = \varphi(d) = \sin 2\beta. \quad (6-10)$$

Equation (6-7) can now be expressed as

$$\left( \frac{\xi^2}{\cos 2\beta} \right) \frac{d^2\varphi}{dz^2} + \varphi = 0 \quad (6-11)$$

and the solution for  $\varphi$ , subject to the boundary conditions Eq. (6-10), is

$$\varphi = \sin 2\beta \left[ \cos \frac{z}{\xi'} + \frac{\left(1 - \cos \frac{L}{\xi'}\right)}{\sin \frac{L}{\xi'}} \cdot \sin \frac{z}{\xi'} \right] \quad (6-12)$$

where

$$\xi' = \sqrt{\frac{\xi^2}{\cos 2\beta}}. \quad (6-13)$$

As we know, sin and cos functions can be approximated by Taylor's series as

$$\begin{aligned} \cos\left(\frac{z}{\xi'}\right) &\approx 1 - \frac{1}{2} \left(\frac{z}{\xi'}\right)^2 \\ \sin\left(\frac{z}{\xi'}\right) &\approx \sin \frac{z}{\xi'}. \end{aligned} \quad (6-14)$$

Therefore Eq. (6-12) can be rewritten as

$$\varphi = \sin 2\beta \left\{ \left[ 1 - \frac{1}{2} \left( \frac{z}{\xi'} \right)^2 \right] + \frac{1}{2} \frac{zL}{\xi'^2} \right\}. \quad (6-15)$$

Substituting  $\varphi$  and  $\xi'$  with  $\theta$  and  $\xi$ , respectively, by Eqs. (6-9) and (6-13), we obtain

$$\theta = \frac{1}{4\xi^2} \sin 2\beta (Lz - z^2). \quad (6-16)$$

As a result of this reorientation, the incident laser light (the  $e$  ray) experiences a  $z$ -dependent refractive index change given by

$$\Delta n = n_{\text{eff}}(\beta + \theta) - n_{\text{eff}}(\beta), \quad (6-17)$$

where  $n_{\text{eff}}(\alpha)$  is the  $e$  ray refractive index:

$$n_{\text{eff}}(\alpha) = \frac{n_e n_o}{(n_e^2 \cos^2 \alpha + n_o^2 \sin^2 \alpha)^{1/2}}. \quad (6-18)$$

Using the approximation

$$(1 + x)^{-1/2} \cong 1 - \frac{1}{2}x \quad \text{for } -1 \leq x \leq 1 \quad (6-19)$$

and introducing  $N$  and  $\Delta\epsilon$  defined as

$$N = \frac{n_o^2 - n_e^2}{n_e^2} = \frac{-\Delta\epsilon}{\epsilon_{\parallel}}, \quad (6-20)$$

$$\Delta\epsilon = \epsilon_{\parallel} - \epsilon_{\perp}$$

Equation (6-18) can be expressed as

$$n_{\text{eff}}(\alpha) \cong n_o \left( 1 - \frac{1}{2} N \sin^2 \alpha \right) = n_o \left( 1 + \frac{1}{2} \frac{\Delta\epsilon}{\epsilon_{\parallel}} \sin^2 \alpha \right). \quad (6-21)$$

Substituting (6-21) into (6-17), we obtain

$$\Delta n = n_o \left( 1 + \frac{1}{2} \frac{\Delta\epsilon}{\epsilon_{\parallel}} \sin^2(\beta + \theta) \right) - n_o \left( 1 + \frac{1}{2} \frac{\Delta\epsilon}{\epsilon_{\parallel}} \sin^2 \beta \right). \quad (6-22)$$

For small  $\theta$ , Eq. (6-22) reduces to

$$\Delta n = \frac{1}{2} \frac{n_{\perp} \Delta \varepsilon}{\varepsilon_{\parallel}} (\sin 2\beta) \cdot \theta. \quad (6-23)$$

Substituting Eq. (6-16) into (6-23), we obtain

$$\Delta n = \frac{n_o \varepsilon_0 (\Delta \varepsilon)^2 \sin^2(2\beta)}{8 \varepsilon_{\parallel} K} (Lz - z^2) \langle E_{\text{op}}^2 \rangle. \quad (6-24)$$

The change in refractive index  $\Delta n$  is proportional to the square of the optical electrical field, i.e.,

$$\Delta n = n_{2(MKS)}(z) \langle E_{\text{op}}^2 \rangle. \quad (6-25)$$

Comparing Eqs. (6-24) with (6-25) the effective  $n_2$  is obtained as

$$n_2(z) = \frac{n_o \varepsilon_0 (\Delta \varepsilon)^2 \sin^2(2\beta)}{8 \varepsilon_{\parallel} K} (Lz - z^2). \quad (6-26)$$

Notice, however, that the  $n_2$  is defined for one  $\mathbf{k}$  vector of the incident light. Also, it is a function of  $z$ , the propagation distance inside the liquid crystal. We should point out that in the above treatment, there is no voltage applied across the liquid crystal, although that is not the case for typical applications. Furthermore, if the light is a beam, i.e., is comprised of an angular spectrum of plane waves, one can only determine an effective  $n_2 = n_{2\text{eff}}$ . The light beam is also expected to change its profile during propagation through the liquid crystal because of the combined effect of diffraction and nonlinearity. In this case, the analysis is even more involved.

## 7 OPTICALLY INDUCED REORIENTATIONAL NONLINEARITY WITH EXTERNAL VOLTAGE

The optically induced molecular reorientation theory introduced in the last section is based on the assumption that the director,  $\mathbf{n}$ , of liquid crystal cell is uniform. This is not the case when an external voltage is applied on the liquid crystal cell, as is commonly the case. Instead, the tilt angle,  $\phi$ , is a function of  $z$  as calculated in Sec. 5. Therefore the angle between the wave vector,  $\mathbf{k}$ , of the incident optical field and the director,  $\mathbf{n}$ , of the liquid crystal is also a function of  $z$ . To handle this new situation, we need to rederive the torque balance equation for liquid crystal cell. For details, see Pea and Banerjee (2001).

As we have seen before, the elastic and electrical torques for the case without an optical field are

$$\begin{aligned}\Gamma_S &= K_{11} \left[ \cos\phi \sin\phi \left( \frac{d\phi}{dz} \right)^2 - \cos^2\phi \frac{d^2\phi}{dz^2} \right] \mathbf{a}_y \\ \Gamma_T &= 0\end{aligned}\quad (7-1)$$

$$\Gamma_B = \left[ K_{33} \left( -\cos\phi \sin\phi \left( \frac{d\phi}{dz} \right)^2 - \sin^2\phi \frac{d^2\phi}{dz^2} \right) \right] \mathbf{a}_y \quad (7-2)$$

$$\Gamma_{\text{ext}} = [-\epsilon_0 \Delta \epsilon E_{\text{ext}}^2 \sin\phi \cos\phi] \mathbf{a}_y.$$

In the equilibrium state, we have

$$\begin{aligned}\Gamma_{\text{ext}} &= -\Gamma_{\text{elastic}} = -(\Gamma_S + \Gamma_T + \Gamma_B) \\ &= \left[ (K_{11} \cos^2\phi + K_{33} \sin^2\phi) \left( \frac{d^2\phi}{dz^2} \right) + (K_{33} - K_{11}) \sin\phi \cos\phi \left( \frac{d\phi}{dz} \right)^2 \right] \mathbf{a}_y.\end{aligned}\quad (7-3)$$

Consider the geometry of interaction as depicted in Fig. 6, where a linearly polarized laser is incident on a homogeneously aligned nematic liquid crystal. The direction of the propagation vector  $\mathbf{k}$  of the laser is  $\beta$  with respect to  $Z$  axis.  $\phi$  is the tilt angle of liquid crystal director and  $\theta$  is the reorientation angle.  $\psi$  is the new tilt angle of liquid crystal director under the influence of optical field and is defined as

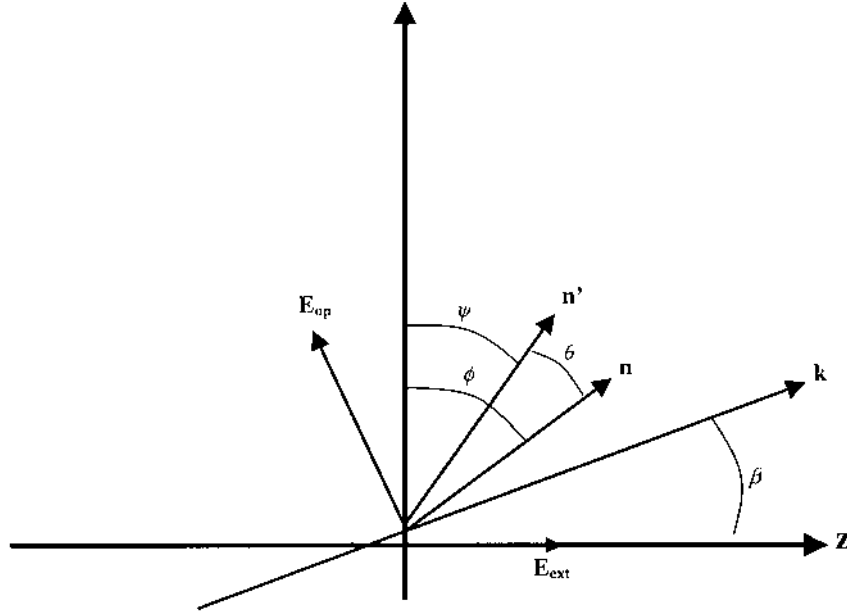
$$\psi = \phi - \theta. \quad (7-4)$$

According this geometry, the original director  $\mathbf{n}$ , the perturbed director  $\mathbf{n}'$ , the optical field  $\mathbf{E}_{\text{op}}$ , and the external electrical field  $\mathbf{E}_{\text{ext}}$  can be expressed in the following form.

$$\begin{aligned}\mathbf{n} &= (\cos\phi(z), 0, \sin\phi(z)) \\ \mathbf{n}' &= (\cos\psi(z), 0, \sin\psi(z)) \\ \mathbf{E}_{\text{op}} &= E_{\text{op}}(\cos\beta, 0, -\sin\beta) \\ \mathbf{E}_{\text{ext}} &= E_{\text{ext}}(0, 0, 1).\end{aligned}\quad (7-5)$$

The revised three molecular fields are obtained by

$$\mathbf{h}_S = K_{11} \left[ -\sin\psi \left( \frac{d\psi}{dz} \right)^2 + \cos\psi \frac{d^2\psi}{dz^2} \right] \mathbf{a}_z$$



**Figure 6** Interaction of a linearly polarized (extraordinary ray) laser with a homogeneously aligned nematic liquid crystal cell under the influence of external electrical field.

$$\mathbf{h}_T = 0 \quad (7-6)$$

$$\mathbf{h}_B = K_{33} \left[ -(\sin^2 \psi + 1) \cos \psi \left( \frac{d\psi}{dz} \right)^2 - \sin \psi \frac{d^2 \psi}{dz^2} \right] \mathbf{a}_x$$

$$- K_{33} \left[ -\sin^3 \psi \left( \frac{d\psi}{dz} \right)^2 \right] \mathbf{a}_z.$$

The new elastic torques, i.e., splay, twist, and bend, are obtained as

$$\Gamma_{S'} = K_{11} \left[ \cos \psi \sin \psi \left( \frac{d\psi}{dz} \right)^2 - \cos^2 \psi \frac{d^2 \psi}{dz^2} \right] \mathbf{a}_y$$

$$\Gamma_{T'} = 0 \quad (7-7)$$

$$\Gamma_{B'} = K_{33} \left[ -\cos \psi \sin \psi \left( \frac{d\psi}{dz} \right)^2 - \sin^2 \psi \frac{d^2 \psi}{dz^2} \right] \mathbf{a}_y.$$

$$\begin{aligned}
\Gamma_{\text{elastic}}' &= \Gamma_S' + \Gamma_T' + \Gamma_B' \\
&= - \left[ (K_{11} \cos^2 \psi + K_{33} \sin^2 \psi) \left( \frac{d^2 \psi}{dz^2} \right) \right. \\
&\quad \left. + (K_{33} - K_{11}) \sin \psi \cos \psi \left( \frac{d\psi}{dz} \right)^2 \right] \mathbf{a}_y.
\end{aligned} \tag{7-8}$$

The optical and new electrical torques can be derived as

$$\Gamma_{\text{op}} = \left[ \frac{\varepsilon_0 \Delta \varepsilon}{2} \right] \langle E_{\text{op}}^2 \rangle \sin(2\beta + 2\psi) \mathbf{a}_y \tag{7-9}$$

$$\Gamma_{\text{ext}}' = \left[ -\frac{\varepsilon_0 \Delta \varepsilon}{2} \right] E_{\text{ext}}^2 \sin(2\psi) \mathbf{a}_y. \tag{7-10}$$

From the torque balance equation, we obtain

$$\Gamma' = \Gamma_S' + \Gamma_T' + \Gamma_B' + \Gamma_{\text{op}} + \Gamma_{\text{ext}}' = \Gamma_{\text{elastic}}' + \Gamma_{\text{op}} + \Gamma_{\text{ext}}' = 0. \tag{7-11}$$

If the reorientation angle  $\theta$  is small, we can assume  $\psi \approx \phi$  and

$$\Gamma_{\text{ext}}' \cong \Gamma_{\text{ext}} = -\Gamma_{\text{elastic}}. \tag{7-12}$$

Then Eq. (7-11) can be rewritten as

$$\Gamma' = \Gamma_{\text{elastic}}' - \Gamma_{\text{elastic}} + \Gamma_{\text{op}}' = 0. \tag{7-13}$$

Substituting Eqs. (7-3), (7-8), and (7-9) into Eq. (7-13), we obtain the differential equation

$$\begin{aligned}
&- \left[ (K_{11} \cos^2 \psi + K_{33} \sin^2 \psi) \left( \frac{d^2 \psi}{dz^2} \right) + (K_{33} - K_{11}) \sin \psi \cos \psi \left( \frac{d\psi}{dz} \right)^2 \right] \mathbf{a}_y \\
&+ \left[ K_{11} \cos^2 \phi + K_{33} \sin^2 \phi \right] \left( \frac{d^2 \phi}{dz^2} \right) + (K_{33} - K_{11}) \sin \phi \cos \phi \left( \frac{d\phi}{dz} \right)^2 \mathbf{a}_y \\
&+ \left[ \frac{\varepsilon_0 \Delta \varepsilon}{2} \langle E_{\text{op}}^2 \rangle \sin(2\beta + 2\psi) \right] \mathbf{a}_y = 0.
\end{aligned} \tag{7-14}$$

Because all torques in Eq. (7-14) are along same direction, we can reduce it to a scalar differential equation. After substituting  $\psi = \phi - \theta$ , Eq. (7-14) can be rewritten as

$$\begin{aligned} & \frac{(K_{11} - K_{33})}{2} \sin(2\phi - 2\theta)(\phi' - \theta')^2 \\ & - [K_{11}\cos^2(\phi - \theta) + K_{33}\sin^2(\phi - \theta)](\phi'' - \theta'') - \frac{(K_{11} - K_{33})}{2} \sin 2\phi \phi'^2 \\ & + (K_{11}\cos^2\phi + K_{33}\sin^2\phi)\phi'' + \frac{\epsilon_0 \Delta \epsilon}{2} \langle E_{\text{op}}^2 \rangle \sin(2\phi + 2\beta - 2\theta) = 0 \end{aligned} \quad (7-15)$$

where  $\phi'$  and  $\theta$  are the first-order derivative of  $\phi$  and  $\theta$  respectively, and  $\phi''$  is the second-order derivative of  $\phi$ .

This is a nonlinear differential equation with two hard-boundary conditions, i.e., the director axis is not perturbed at the boundary ( $\theta = 0$  at  $z = 0$  and  $z = L$ ). Equation (7-15) is similar with Eq. (6-5). Both equations describe the reorientation  $\theta$  of the liquid crystal director caused by the external optical plane wave. But there are important differences between them.

- (1) Equation (7-15) is more complex than Eq. (6-5).
- (2) In Eq. (6-5), the tilt angle  $\phi$  of the liquid crystal director is equal 0. But in Eq. (7-15),  $\phi$  is a function of  $z$  due to the influence of external electrical field. Therefore  $\phi'$  and  $\phi''$  are the functions of  $z$  too.
- (3) In Eq. (6-5),  $E_{\text{op}}$  is usually assumed to be a constant. But in Eq. (7-15),  $E_{\text{op}}$  can be a function of  $z$ .

As a result of this reorientation, the incident laser beam (the  $e$  ray) experiences a  $z$ -dependent refractive index change given by

$$\Delta n(z) = n_{\text{eff}}\left(\frac{\pi}{2} - \phi(z) - \beta + \theta(z)\right) - n_{\text{eff}}\left(\frac{\pi}{2} - \phi(z) - \beta\right) \quad (7-16)$$

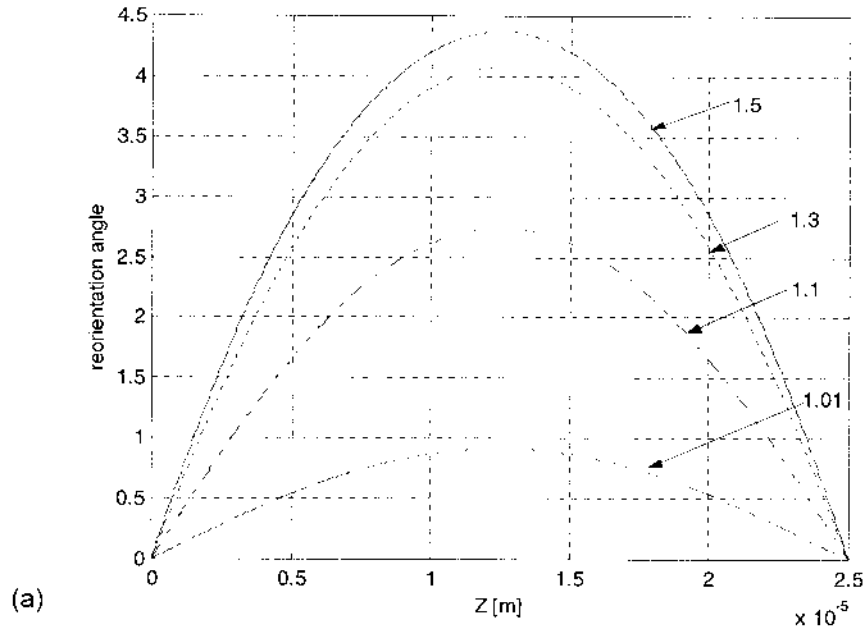
where  $n_{\text{eff}}(\alpha)$  is the  $e$  ray refractive index and is defined as:

$$n_{\text{eff}}(\alpha) = \frac{n_e n_o}{(n_e^2 \cos^2 \alpha + n_o^2 \sin^2 \alpha)^{1/2}} = n_o (1 + N \sin^2 \alpha)^{-1/2} \quad (7-17)$$

where  $N = \frac{n_o^2 - n_e^2}{n_e^2}$ , and  $\alpha$  is the angle between  $\mathbf{k}$  and  $\mathbf{n}$ . Substituting Eq. (7-17) into Eq. (7-16), we obtain

$$\Delta n(z) = n_o [(1 + N \cos^2(\phi(z) + \beta - \theta(z)))^{-1/2} - (1 + N \cos^2(\phi(z) + \beta))^{-1/2}]. \quad (7-18)$$

To solve this nonlinear boundary value problem, a *finite difference method* modified with *Newton's iteration* is used here [Sewell,1988,]. For all simulations,  $n_o = 1.52$ , and  $n_e = 1.746$ . Fig. 7(a) and (c) are the numerical results of  $\theta$  for  $V/V_0 = 1.01, 1.1, 1.2, 1.4, 1.5$  and for  $V/V_0 = 1.4, 1.5, 1.8, 2.4, 4$ , respectively, with  $E_{op} = 1 \times 10^5$  V/m and  $\beta = 0$ , where  $V_0$  is the threshold voltage of liquid crystal cell. Fig. 7(b) and (d) are the  $\Delta n$  corresponding to the reorientation angle shown in Fig. 7(a) and (c) obtained by Eq. (7-18). In Fig. 7(a) and (b),  $\theta$  and  $\Delta n$  increase while the external voltage increases. They reach the peak at  $V/V_0 = 1.4$ . When  $V/V_0$  is greater than 1.4,  $\theta$  and  $\Delta n$  decrease while the external voltage increases, as shown in Fig. 7(c) and (d). This agrees with our expectation that when the external voltage is much higher than the threshold voltage, the director is strongly tilted along the external electrical field and is hard to be reoriented by the external optical field. Note also from the previous section (no applied voltage) that  $\Delta n = 0$  if



**Figure 7** (a) Reorientation  $\theta$  of liquid crystal director for  $V/V_0 = 1.01, 1.1, 1.2, 1.4, 1.5$  with  $E_{op} = 1 \times 10^5$  V/m and  $\beta = 0$ . (b)  $\Delta n$  of liquid crystal for  $V/V_0 = 1.01, 1.1, 1.2, 1.4, 1.5$  with  $E_{op} = 1 \times 10^5$  V/m and  $\beta = 0$ . (c) Reorientation  $\theta$  of liquid crystal director for  $V/V_0 = 1.5, 1.8, 2.5, 4.0$  with  $E_{op} = 1 \times 10^5$  V/m and  $\beta = 0$ . (d)  $\Delta n$  of liquid crystal for  $V/V_0 = 1.5, 1.8, 2.5, 4.0$  with  $E_{op} = 1 \times 10^5$  V/m and  $\beta = 0$ .

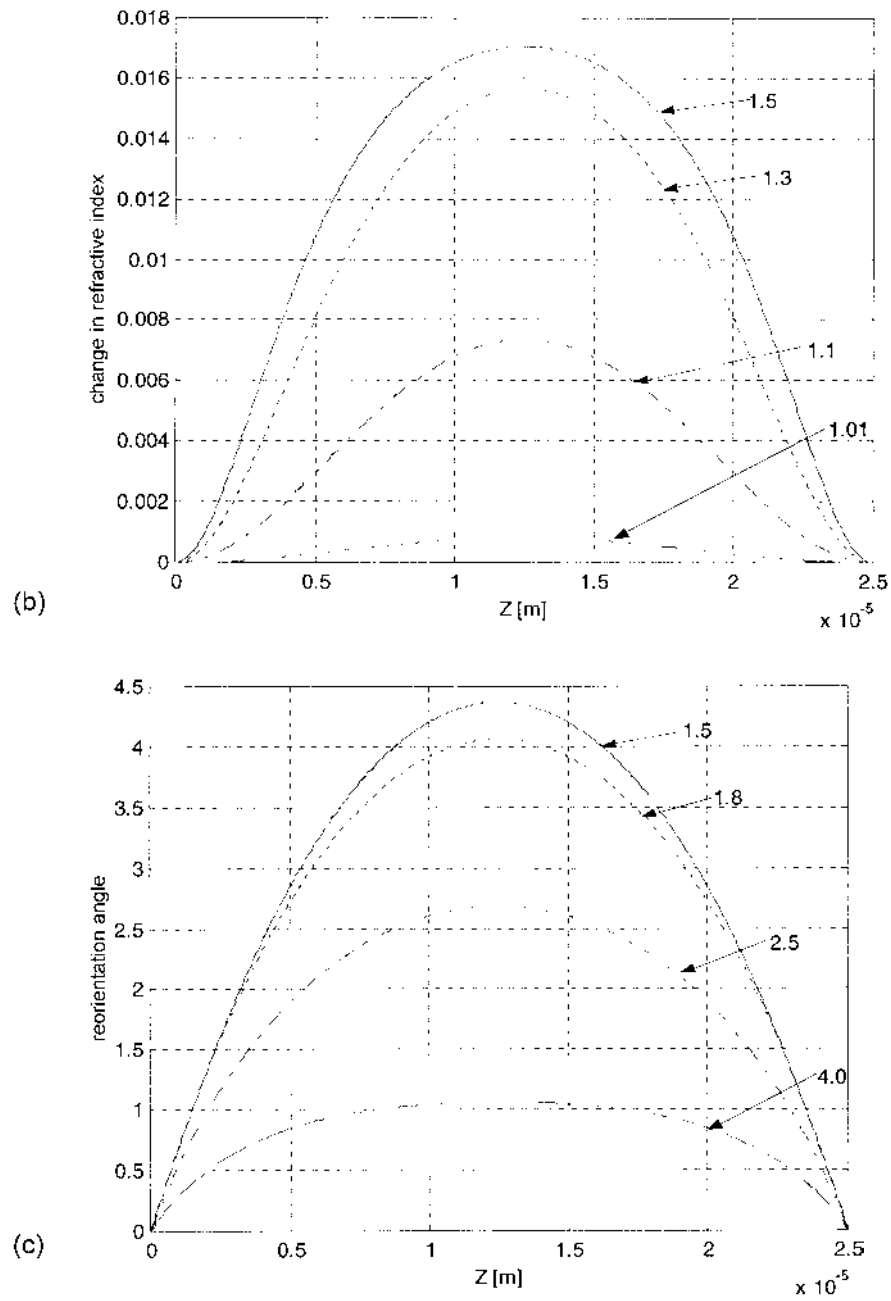
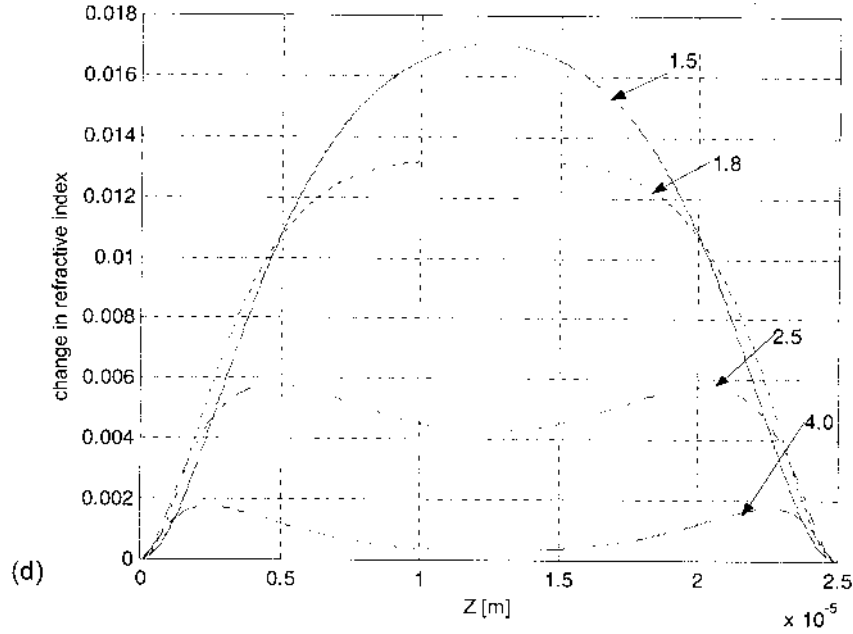


Figure 7 Continued.



**Figure 7** Continued.

$\beta = 0$ . In our case here,  $\Delta n$  is still present because the director is tilted by application of the external field  $\mathbf{E}_{\text{ext}}$ .

We can also compare the reorientation  $\theta$  and the change of refractive index  $\Delta n$  for different optical intensities. This is assigned as a problem at the end of the chapter; we will briefly summarize the results here. The maxima of  $\theta$  and  $\Delta n$  are approximately linearly proportional to optical intensity linearly for a range of optical intensities, typically within  $10^5 \text{ W/m}^2$ . For higher optical intensities ( $I > 10^7 \text{ W/m}^2$ ), this linear relationship does not exist anymore, and the peak values of  $\theta$  and  $\Delta n$  tend to saturate.

## 8 ANALYSIS OF BEAM PROPAGATION IN LIQUID CRYSTALS AND z-SCAN

In the last section, we have seen how the induced refractive index as a result of reorientational nonlinearity is a function of position in the crystal and depends on the intensity of the incident optical plane wave, its angle with

respect to the direction of propagation, and on the applied voltage. In this section, we will model the propagation of optical beams, viz., Gaussian beams through a liquid crystal sample in the presence of an applied voltage. In this connection, we will develop the  $z$ -scan method, introduced in Chap. 4, which can be used as a characterization tool for finding the effective nonlinearity of the liquid crystal [Pea and Banerjee (2001)].

Because the induced refractive index is not a constant, we will have to analyze optical propagation using a split-step beam propagation method, which is summarized in Appendix A. To simplify the calculation, we choose to simulate a one-transverse-dimensional case. However, this treatment does not compromise the significance of the simulation. Because of the circular symmetry for the Gaussian beam, the nonlinear effect in the other transverse dimension is quite similar to that in the one-dimensional simulation. Two-dimensional simulations will be straightforward and similar to the one-transverse case except for the requirement of much more intensive computation.

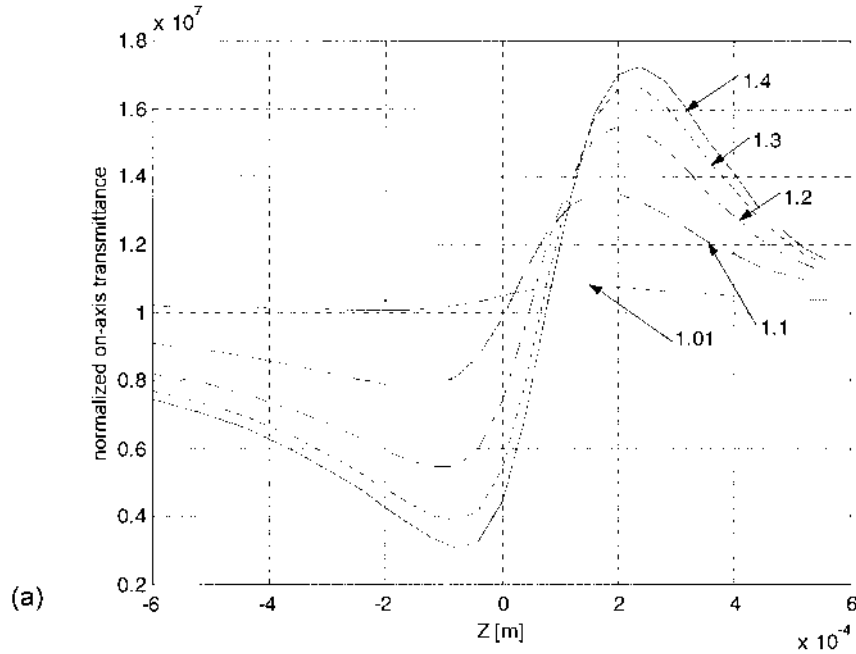
The nonlinear  $\Delta n$ , which is used in the split-step method, is similar with the  $\Delta n$  discussed in Sec. 7. However, note that calculation of  $\Delta n$  in Sec. 7 given by Eq. (7-18) is based on the assumption that the incident light is a plane wave with constant amplitude. This assumption is not valid anymore because the incident light is a (focused) Gaussian beam in our case. To apply the optically induced reorientation theory derived in Sec. 7 to a Gaussian beam case, we decompose the Gaussian beam into its angular plane wave spectrum using *Fourier transform*. The components of the angular spectrum can be identified as plane waves traveling in different directions. According to the theory in Sec. 7, each component plane traveling in the direction  $(k_x/k_0)$  in the paraxial approximation wave causes its own  $\Delta n(k_x)$ . Consequently, the total  $\Delta n(x)$  caused by the Gaussian beam is the *inverse Fourier transform* of  $\Delta n(k_x)$ .

To evaluate  $\Delta n(k_x)$ , we need to solve  $\theta$  in Eq. (7-15) for each component plane wave of the angular spectrum. Before doing this, we need to consider the amplitude of the optical plane waves. In Sec. 7, we assume that the amplitude of plane wave is constant. But because of the nature of the *Gaussian* beam, the amplitude of each component plane waves does not keep constant over the sample. However, the solution of Eq. (7-15) needs the precise variation of  $E_{\text{op}}(x, z)$ . We overcome this difficulty by assuming that the envelope of  $E_{\text{op}}(x, z)$  is not too different from the original Gaussian beam because the thickness of sample is very small compared with the focal length of the external lens in the  $z$ -scan setup. Therefore we use the initial Gaussian  $E_{\text{op}}(x, z)$  as the initial guess to solve for  $\theta$  in the Eq. (7-15) and obtain the first iterated value of  $\Delta n(k_x, z)$  using Eq. (7-18). By using this  $\Delta n(k_x, z)$  in the split-step method, the optical field  $E_{\text{op}}(x, z)$  everywhere inside the sample can be

obtained. By using this new  $E_{op}(x,z)$ , we re-solve  $\theta$  in Eq. (7-15) and calculate the  $\Delta n(k_{x,z})$  using Eq. (7-18) again. We iterate this procedure until the final optical field emerging from the sample converges to within a proper limit.

After we find the final optical field emerging from the sample, the far-field beam profile can be obtained by applying the linear diffraction theory. According to the *Fresnel diffraction formula*, the optical wave propagating through a distance  $z$  in the medium can be found by taking the inverse Fourier transform of the product of the spatial transfer function and angular spectrum of initial optical field. The simulation results are shown in Fig. 8(a–c). Fig. 8(a,b) shows the  $z$ -scan graphs with the applied voltage as a parameter, while Fig. 8(c) shows the effective nonlinear refractive index coefficient calculated on the basis of the slope of the  $z$ -scan graphs, as discussed in Chap. 4.

Fig. 9(a) shows typical experimental results for transmitted power vs. sample position in a  $z$ -scan experiment for a liquid crystal sample for ap-



**Figure 8** (a)  $z$ -scan simulation results for  $f = 50.2 \text{ mm}$  and  $V/V_0 = 1.01, 1.1, 1.2, 1.3, 1.4$  with  $E_{op} = 2 \times 10^5 \text{ V/m}$ . (b)  $z$ -scan simulation results for  $f = 50.2 \text{ mm}$  and  $V/V_0 = 1.4, 1.8, 2.2, 2.6, 4$  with  $E_{op} = 2 \times 10^5 \text{ V/m}$ . (c) Calculated  $n_{2-eff}$  for  $f = 50.2 \text{ mm}$  with  $E_{op} = 2 \times 10^5 \text{ V/m}$ .

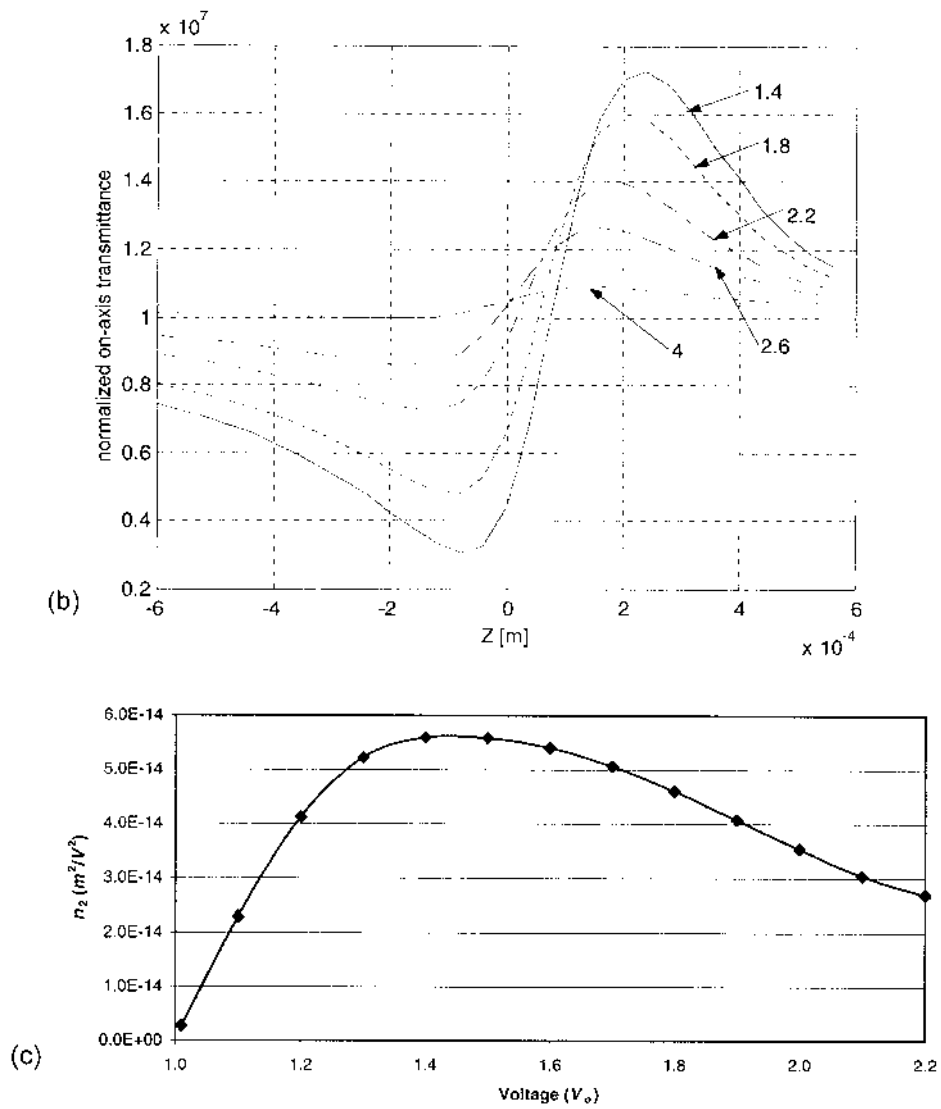
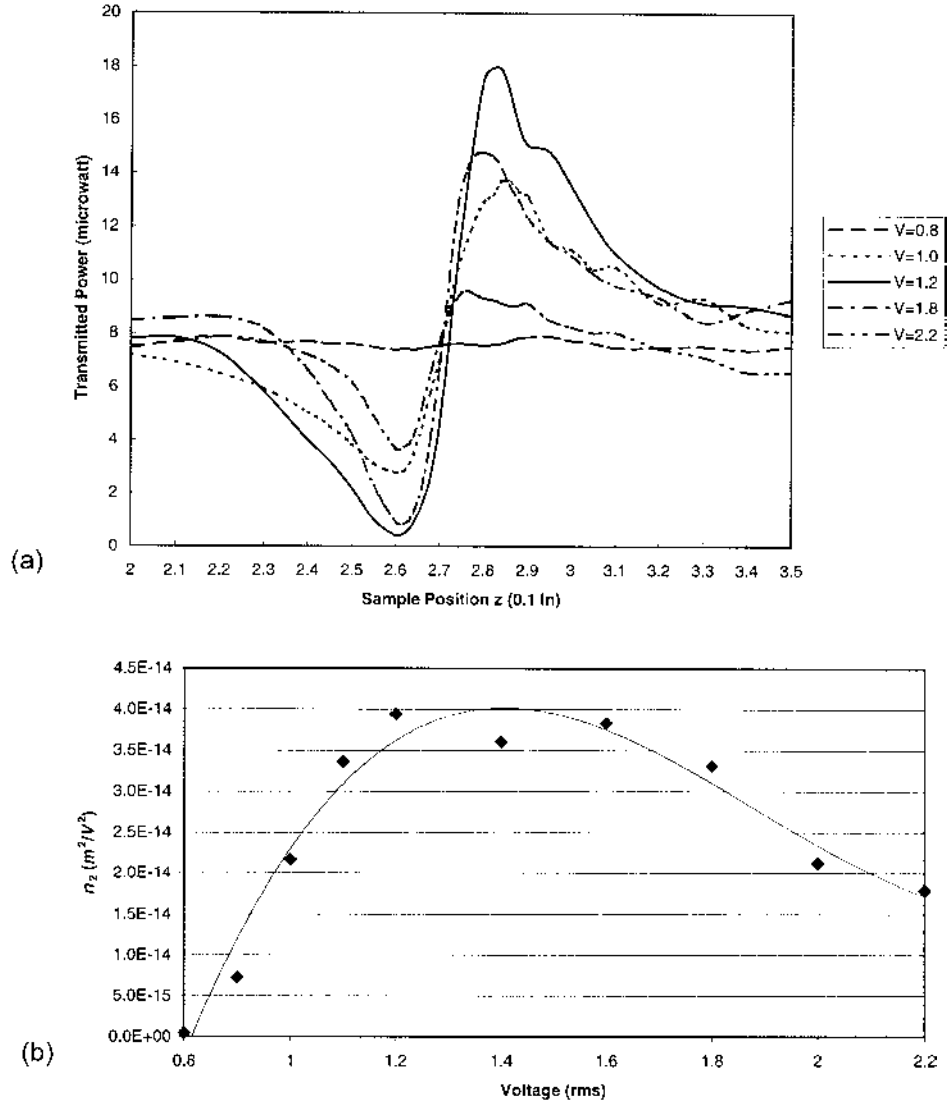


Figure 8 Continued.



**Figure 9** (a) Measured  $z$ -scan graph for  $f = 50.2$  mm and aperture diameter = 1 mm. (b) Experimental  $n_{2-eff}$  for  $f = 50.2$  mm with aperture diameter = 1 mm.

plied voltages 0.8, 1.0, 1.2, 1.8, and 2.2 V. In this case, the focal length is 50.2 mm and aperture diameter on the observation plane is 1 mm. The scan range is from 0.2 to 0.35 in. The scan step,  $\Delta z$ , is varied from about 0.002 to about 0.01 in., depending on the position of the sample. This is performed so as to obtain the maximum information around the back focal plane of the external lens where the measurement values are to be used to find the slope of the  $z$ -scan graph, and hence the effective nonlinear refractive index coefficient. The curves of 1.0, 1.2, and 1.8 V have the typical valley-peak  $z$ -scan pattern. The curves of 0.8 and 2.2 V do not exhibit the significant peak-valley pattern. Fig. 9(b) is the experimental  $n_{2\text{-eff}}$  with external voltage  $V_{\text{rms}} = 0.8, 0.9, 1.0, 1.1, 1.2, 1.4, 1.6, 1.8, 2.0, \text{ and } 2.2 \text{ V}$  for an aperture diameter of 1 mm.

As seen both from Figs. 8(c) and 9(b), the  $n_{2\text{-eff}}$  increases and reaches a maximum at  $V/V_0 = 1.4$  before starting to decrease. Simulated and experimental  $n_{2\text{-eff}}$  have similar trend. Furthermore, there is reasonable agreement between the calculated and measured values of  $n_{2\text{-eff}}$ .

## 9 PROBLEMS

1. Assume that a nonlinear optical material has a refractive index that depends on the optical intensity, and on the angle  $\theta$  the plane wave makes w.r.t. the  $Z$  axis, the nominal direction of propagation. In the paraxial approximation, assume that the angular dependence is of the form  $\cos \theta$ . Determine how an initial optical profile of the form  $E_i = E_0 (1 + m \cos Kx)$ ,  $m \ll 1$  will propagate through the medium. Does one expect to observe any periodic focusing of the initial profile, as seen in the case of linear propagation (Talbot imaging)?
2. Plot the reorientation angle and the resulting change of refractive index as a function of position inside a liquid crystal sample with the optical intensity as a parameter. Find the range of input intensities over which the change in the peaks of the reorientation and induced refractive index are approximately linear with respect to the change in input intensity. Take  $\beta = 0$ , and input intensities in multiples of  $I_0 = 1.326 \times 10^5 \text{ W/m}^2$ . Also, take  $V = 1.5V_0$ . For what value of intensity does the induced refractive index change saturate?
3. Plot the reorientation angle and the resulting change of refractive index as a function of position inside a liquid crystal sample with the angle  $\beta$  between the optical wave and the  $Z$  axis as a parameter. Use  $\beta = -20^\circ, -10^\circ, 0^\circ, 10^\circ, 20^\circ$  with external voltage  $V = 1.5V_0$

and  $E_{op} = 1 \times 10^5$  V/m. Are the plots symmetrical about  $\beta = 0$ ? Give physical reasons for your answers.

4. Design a liquid crystal based optical limiter that comprises a lens and a liquid crystal sample placed around its back focal plane. With the bias across the sample as a parameter, plot the on-axis transmittance as a function of the input power.

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# 11

## Self-Organization in Nonlinear Optical Systems

### 1 INTRODUCTION

*Self-organization* refers to the development of spatial and/or temporal patterns during the temporal response of a nonlinear system to a given input. Almost always, self-organization results from symmetry breaking in the system. Many natural phenomena as well as the brain or animal behavioral patterns exhibit self-organization. The convective rolls in a liquid when it is heated beyond the *Rayleigh–Bernard instability* point are an example of pattern formation in *hydrodynamics* due to self-organization. Here, the temperature difference is the driving force or input parameter. Below the critical or threshold temperature, one can only observe the random motion of the liquid particles. Besides, in an open container containing the fluid, surface tension can also affect the flow, causing tessellation of the surface and formation of *hexagonal cells*. As stated above, such spontaneous pattern formation is exactly what is termed self-organization, but there is no agent inside the system that does the organizing. The motion of the whole is no longer the sum of the motion of the parts due to the nonlinear interactions between the parts and the environment. Another example of pattern formation is a “wave” among spectators in a stadium—individual spectators communicate and cluster together in groups to create a nearly synchronized pattern that spreads throughout the stadium. Speaking of which, there is enough evidence that human behavioral patterns are self-organized. The human body, for example, is a complex system comprising about  $10^2$  joints,  $10^3$  muscles,  $10^3$  cell types, and  $10^{14}$  neurons or neuron connections. The actions of communication, body movement, etc., are the results of self-organization of this complex system pertaining to a certain control or input parameter (stimulus). In a similar fashion, it has been shown that the brain itself is an active, dynamical self-organizing system.

For more on the self-organizing aspects of the brain and human behavior, the readers are referred to Scott Kelso (1995), Kohonen (1984), and Haken (1983).

Some of the elementary concepts and conditions for self-organization are as follows (see Scott Kelso, 1995).

1. Patterns arise spontaneously as a result of nonlinear coupling between large numbers of interacting components.
2. The system must be far from equilibrium. Due to nonlinear interactions, energy is not distributed evenly but coalesce into patterns or flows.
3. Relevant degrees of freedom, or order parameters, must exist near nonequilibrium phase transitions, where loss of stability gives rise to new patterns and/or switching between patterns.
4. Noise must be present in the system, so that fluctuations can “feel” the system stability and provide for the system to self-organize into different patterns.

In this chapter, we will discuss self-organization and its effects in optics. In fact, one of the most exciting and potentially useful areas of current research in optics involves the understanding and exploitation of self-organization in nonlinear optical systems. This self-organization may sometimes lead to the evolution of complex spatial patterns which can be regarded as the *nonlinear eigenmodes* of the system. Generation of these patterns is characteristically marked by the presence of intensity thresholds. The detailed study of the self-organization process, including the spatio-temporal evolution, is needed in order to harness these effects for potential practical applications.

For a long time in nonlinear optics, only problems of temporal dynamics were investigated. However, spatial distributions were only assumed, without regard to their time evolution and hence relationship with temporal instabilities. However, in a nonlinear system with complicated temporal dynamics, it turns out that one cannot retain purity in spatial dimensionality. It is therefore equally important to investigate the dynamics of the transverse spatial variations which in fact give rise to very interesting patterns due to self-organization. A vast wealth of patterns can be achieved by using a nonlinear optical element with feedback that has the capability to provide for field transformation, e.g., by spatial filtering. These types of systems have been called *optical kaleidoscopes* simply because of the different self-organized patterns that they can generate. Examples of nonlinear self-organized kaleidoscopic patterns are *rolls*, *rotatory waves*, *optical spirals*, *hexagonal patterns*, patterns with more complicated geometry, pattern hopping, etc. An excellent reference for this as well as self-organization in

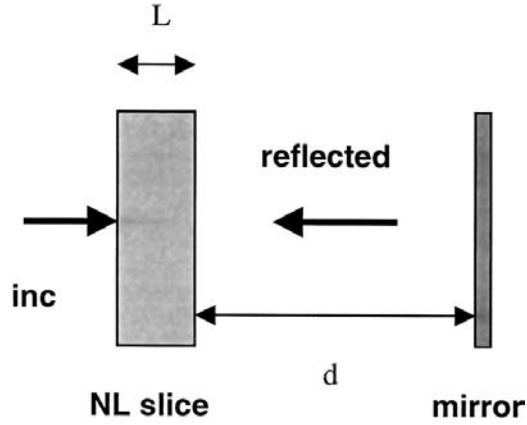
different nonlinear optical systems is the book by Vorontsov and Miller (1995).

Information processing applications of nonlinear optics are closely linked to the ability to control nonlinear optical systems which can self-organize in different ways. For instance, different patterns formed through self-organization can be used for coding and processing of optical information (Vorontsov and Miller, 1995). It has been proposed that the existence of several modes in a laser can be used as a base for *synergetic computing* (Fuchs and Haken, 1988). Fourier filtering techniques have been used in conjunction with nonlinear optical systems for information processing. Degtiarev and Vorontsov (1995) used Fourier filtering in the path of an LCLV system with feedback for *phase distortion suppression*. Such nonlinear optical systems with feedback have also been used for various kinds of pattern generation such as hexagons, rolls, etc. (Vorontsov and Firth, 1994). The dynamics of pattern formation in a coupled LCLV system with feedback has been studied by Thuring et al. (1996).

Photoinduced scattering of laser radiation into self-organizing patterns has been observed over the past several years in a substantial number of nonlinear materials including gases and liquids (Grynberg et al., 1988; Pender and Hesselink, 1990; Thuring et al., 1993; Tamburrini et al., 1993; Grynberg et al., 1994; Gluckstad and Saffman, 1995). Among solids, photorefractive (PR) materials such as  $\text{KNbO}_3$  have been observed to exhibit a rich variety of such scattering including hexagonal pattern formation and rotation, as well as other patterns depending on the experimental conditions (Honda, 1993; Banerjee et al., 1995). Furthermore, simultaneous pattern generation and *self-phase conjugation* have been observed due to self-organization in this material under other conditions (Kukhtarev et al., 1995; Honda, 1995; Banerjee et al., 1996; Honda et al., 1997; Denz et al., 1998). Hexagon formation has also been observed in other PR materials as well, such as  $\text{BaTiO}_3$  (Honda and Matsumoto, 1995; Uesu et al., 1998).

## 2 NONLINEAR TWO-DIMENSIONAL SYSTEMS WITH FEEDBACK

A classic example of a two-dimensional nonlinear system with feedback is a slice of a *Kerr medium* with a feedback mirror, as shown in Fig. 1 (Firth, 1995). For simplicity, assume that the *Kerr slice* is antireflection-coated. Consider a plane wave incident on the Kerr slice. The feedback mirror generates a counterpropagating wave in the Kerr slice and closes the feedback loop. No energy is stored in the system, and the complexity arises from the cross-phase modulation of the two waves in the Kerr slice.



**Figure 1** Schematic diagram of a nonlinear system with feedback in which the nonlinear slice is a Kerr medium.

The Kerr slice can be modeled by a time- and space-dependent refractive index  $\Delta n$  which also nonlinearly depends on the optical field inside the material (Firth, 1995):

$$\tau \partial \Delta n / \partial \tau + \Delta n = l_D^2 \nabla^2 \Delta n + n_2 |E|^2. \quad (2-1)$$

In Eq. (2.1),  $\tau$  represents a characteristic *time constant* and  $l_D$  is a characteristic *diffusion length*. In the steady state and with no diffusion, the refractive index reverts to the traditional expression derived in Chap. 4. The total optical field in the material can be regarded as the sum of a forward and backward propagating field:

$$E = 1/2 [C_e \exp j(\omega_0 t - k_0 z) + C_e' \exp j(\omega_0 t + k_0 z) + c.c.], \quad (2-2)$$

where  $C_e$  and  $C_e'$  are slowly varying envelopes of the forward and backward propagating waves. Now assuming

- (a) No absorption in the Kerr material
- (b) A thin slice of the Kerr material so that we can neglect diffraction of the optical field

the evolution of the forward and backward propagating optical field envelopes in the material can be canonically expressed in the form (Firth, 1995)

$$\partial C_e / \partial \bar{z} = -j\chi \delta \bar{n}, \quad \partial C_e' / \partial \bar{z} = j\chi \delta \bar{n}, \quad (2-3)$$

with the normalized change in refractive index  $\delta\bar{n}$  obeying the evolution equation

$$\tau\partial\delta\bar{n}/\partial\bar{t} + \delta\bar{n} = l_D^2 \nabla_{\perp}^2 \delta\bar{n} + |C_e|^2 + |C_e|^2. \quad (2-4)$$

The parameter  $\chi$  determines the strength of the nonlinearity, and its sign (+ or -) dictates whether the material is focusing or defocusing, respectively.

Finally, outside the medium, the evolution of the optical fields is determined by the paraxial equation for propagation in free space (see Chap. 1):

$$2jk_0\partial C_e/\partial z = \nabla_{\perp}^2 C_e. \quad (2-5)$$

As shown by Firth (1995), Eqs. (2.3)–(2.5) admit a steady-state solution of the form

$$|C_e|^2 = I_0, |C_e|^2 = RI_0, \delta\bar{n} = I_0(1 + R), \quad (2-6)$$

where  $R$  is the reflectivity of the mirror and  $I_0$  is the intensity of the forward or pump field. In order to determine the stability of the solutions, one can introduce perturbations of the form  $\exp[j(\Omega t - \mathbf{K} \cdot \mathbf{R})]$  where  $\Omega$  and  $\mathbf{K}$  denote the oscillation frequency and the wavevector of the perturbations. Upon substitution into Eqs. (2.3)–(2.5) along with the boundary conditions, one can find the following condition that needs to be satisfied for a stable solution:

$$1 + K^2 + j\Omega = 2RI_0\chi \sin(\sigma K^2) \exp -j\Omega t_R \quad K = |\mathbf{K}| \quad (2-7)$$

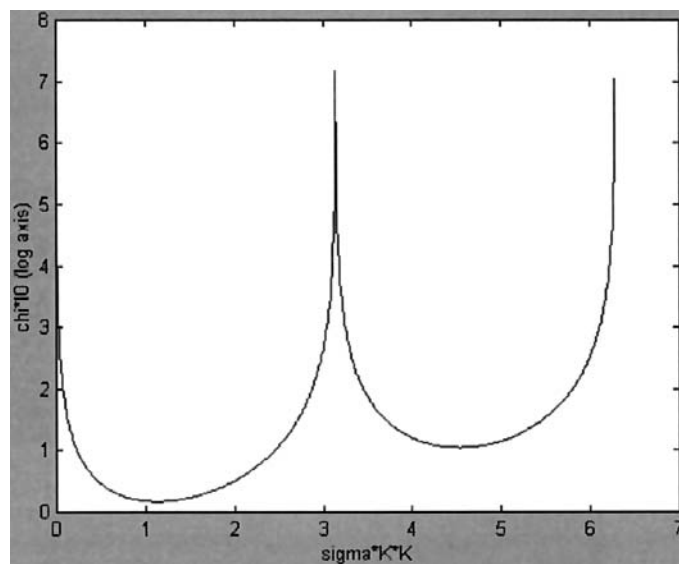
where  $t_R$  is the round trip time between the Kerr slice and the feedback mirror and  $\sigma = d/k_0 l_D^2$  is a dimensionless parameter which is the ratio of the relative strength of diffraction and diffusion.

Limiting ourselves to the simpler case of  $\Omega=0$ , one can obtain the instability threshold for a non-oscillating perturbation as

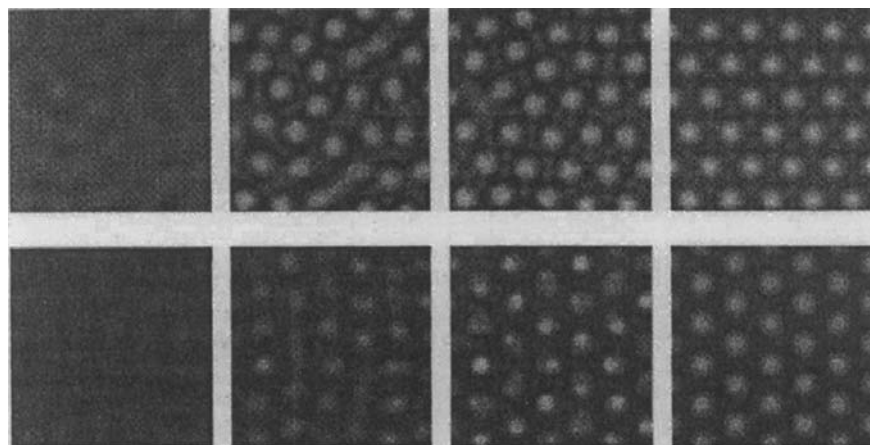
$$\chi I_0 = (1 + K^2)/2R \sin(\sigma K^2). \quad (2-8)$$

Equation (2.8) is plotted as a function of  $\sigma K^2$  in Fig. 2. For small diffusion  $\sigma \gg 1$ , the minimum instability threshold is at  $\sigma K^2 \approx \pi/2$  for  $\chi > 0$  (focusing medium). For smaller  $\sigma$ , the threshold increases while the minima move to smaller values of  $\sigma K^2$ . Similar deductions can be made for defocusing media.

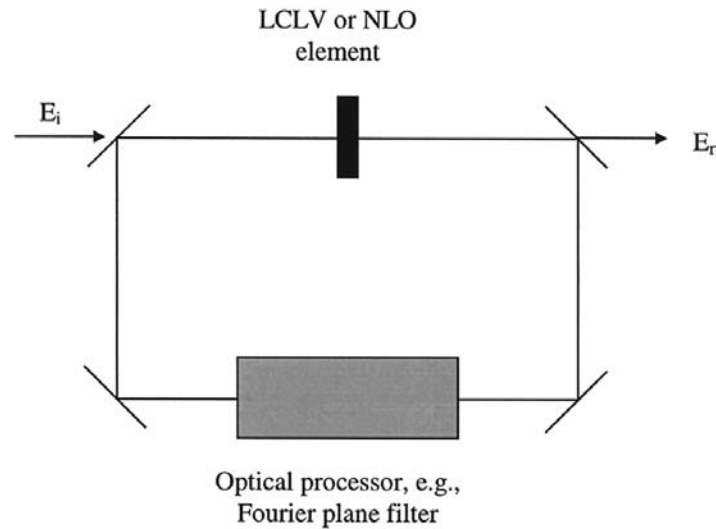
The results above show the regions of linear stability (or instability) but do not predict which of the  $\mathbf{K}$  vectors actually can contribute to pattern formation. To study this, one needs a more detailed nonlinear analysis. We will defer the discussion of kind of analysis till later in the chapter. However, rigorous numerical analysis is also required to analyze the spatio-temporal pattern formation due to self-organization. Initial conditions are a random perturbation of the steady-state uniform solution. As shown in Fig. 3, under



**Figure 2** Threshold curve for a focusing medium for  $R = 0.99$  and  $\sigma = 1$ .



**Figure 3** Backward field intensity in a focusing (top) and defocusing (bottom) medium. Time increases from left to right. The reflection coefficient  $R = 0.9$ , and the round trip time is 0.05. [Firth (1995)].



**Figure 4** Typical optical feedback scheme employing unidirectional cavity with liquid crystal light valve (LCLV) or nonlinear optical element along with optical processor in the feedback path.

certain conditions, the random noise grows until bright spots appear and arrange themselves on a hexagonal grid. The simulation results are in agreement with experimental observations using Kerr slices (Tamburrini et al., 1993).

Other more elaborate nonlinear feedback systems employ a nonlinear material or a *liquid crystal light valve* in a unidirectional ring cavity. A typical schematic is shown in Fig. 4. In this way, additional optical elements can be inserted in the feedback path to enhance pattern formation or, in some cases, suppress *phase distortion*. For instance, a Fourier plane filter can be placed in the feedback loop to facilitate suppression of a range of spatial frequencies where, otherwise, amplification of the phase would be observed after a round trip. Applications of this kind of arrangement can be found in Degtiarev and Voronotsov (1995).

### 3 SELF-ORGANIZATION IN PHOTOREFRACTIVE MATERIALS

In the remainder of this chapter, we will discuss self-organization leading to pattern formation in PR materials. The effect can be heuristically explained as occurring due to a *photoinduced holographic scattering* which develops in

two stages (Banerjee et al., 1995). In the first stage, scattered light is rearranged into a cone which corresponds to a Fabry–Perot mode of the nonlinear cavity formed by the refractive index mismatch at the crystal interfaces. *Reflection gratings*, sometimes aided by *transmission gratings*, may nonlinearly modify the cavity characteristic and the cone angle. At the second stage, waves scattered in the cone write new holographic gratings (second-generation gratings), and those among them that have holographic grating vectors equal to the strongest gratings from the set of first-generation gratings are enhanced, following a winner-take-all route. This holographic self-organization model conceptually explains the appearance of a hexagonal spot structure around the transmitted beam. Other heuristic explanations are based on the *Talbot effect*; this was enunciated by Tamburrini et al. (1993) for a liquid crystal and extended to the case of  $\text{KNbO}_3$  by Honda and Matsumoto (1995). Other simplified explanations of hexagon formation also exist in the literature (Firth, 1995). In this case, the authors use a very simplified although, maybe, unrelated model of nonlinear susceptibility in the understanding of hexagonal pattern formation in photorefractives. The detailed physics of hexagonal pattern generation in photorefractives in our opinion is complicated and not yet well understood.

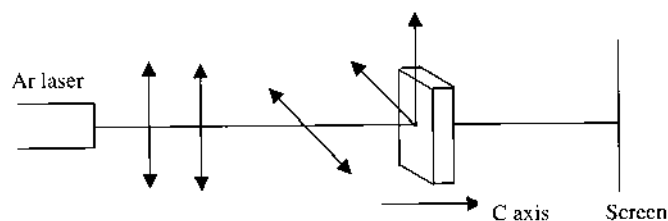
$\text{KNbO}_3$  is a biaxial electro-optic material with orthorhombic symmetry and has excellent PR properties marked by large beam coupling gains (Medrano et al., 1988), fast buildup times, and large anisotropies (Voit et al., 1987) (see Chap. 9 for details on PR nonlinear optics). Furthermore, Fe doping in  $\text{KNbO}_3$  is known to increase the maximum value of the two-beam coupling gain (Medrano et al., 1994).  $\text{KNbO}_3$ -based *phase conjugators* have been implemented in various configurations (Medrano et al., 1994), and material properties of the crystal have been extensively studied (Zgonik et al., 1995). The net optical nonlinearity of  $\text{KNbO}_3$  has been studied using standard *z-scan* techniques (see Chap. 4) (Song et al., 1993). Electrical measurements have been also performed and give valuable information about the *Maxwell relaxation time*, *screening length*, and *photogalvanic current* (Noginova et al., 1997).

Self-organization of an Ar laser single-beam scattering in a PR  $\text{KNbO}_3\text{:Fe}$  crystal, first into a scattering cone, and then into a hexagonal pattern, was observed by Honda (1993) and Banerjee et al. (1995). Furthermore, these spots may be made to rotate about the center, and the rotation speed depends on the misalignment of the incident beam from the *c*-axis and the power of the beam. The hexagonal pattern is also influenced in real time by a low-power He–Ne laser (wavelength 632 nm): the spot pattern erases in about a second after the He–Ne laser is turned on, leaving only the scattering cone, and reappears a second after the He–Ne is turned off.

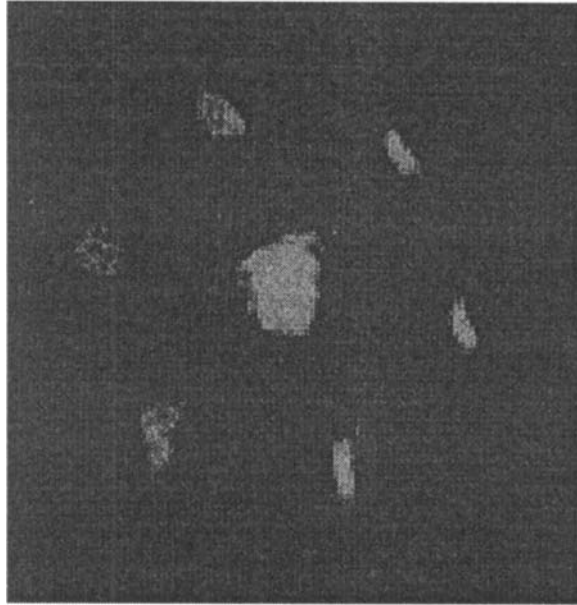
In the simplest experimental setup, an Ar laser (wavelength 514 nm) with horizontal polarization and with initial beam diameter 1 mm is reduced to a beam diameter of 0.5 mm using a confocal lens combination and illuminates a  $\text{KNbO}_3\text{:Fe}$  crystal of dimensions  $6 \times 6 \times 7 \text{ mm}^3$  (see Fig. 5). (A slightly converging beam may also be used.) When the beam is normal to the incident surface, the far-field pattern is stationary in time and comprises a strong central spot with a peripheral ring which appears instantaneously, and thereafter evolves into six symmetrically spaced spots on the scattering cone (see Fig. 6a). This far-field pattern is observed simultaneously both in the forward and backward directions; however, the diffraction efficiencies (discussed in more detail below) are not identical.

The semi-angle of divergence  $\theta$  of the peripheral cone is approximately  $0.8^\circ$  in air and is independent of the incident power. The time taken to form the spots is a few seconds for an incident power of 7.5 mW, although the spots may be formed for lower incident powers as well, with a longer formation time. The ring and all spots (central and peripheral) are also predominantly horizontally polarized. The diffraction efficiency for the spots in the forward direction is large: the intensity ratio of each transmitted peripheral spot to the transmitted central spot, which we term the forward diffraction efficiency per spot, is over 7%, for a total forward scattering efficiency into all six spots of 42%. The corresponding diffraction efficiency in the backward direction is about 4% per spot. Finally, the diffraction efficiencies seem to be relatively independent of the incident power over the range of powers investigated (7.5–30 mW).

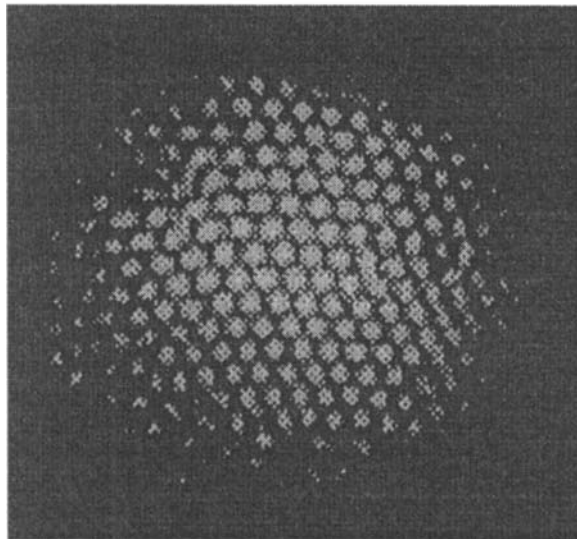
Upon imaging different planes in the crystal (including the exit face) by a lens for the sake of visualization of the transverse nature of the optical fields, we have found, as shown in Fig. 6b, a periodic transverse hexagonal pattern at approximately the exit face of the crystal. Moreover, when the crystal is moved longitudinally by 0.5 cm, the same transverse pattern is re-



**Figure 5** Experimental setup to observe hexagon formation in potassium niobate. The two vertical arrows immediately behind the laser denote the positions of collimating lenses.



(a)



(b)

**Figure 6** (a) Far field transmission pattern showing central spot and hexagonal pattern, and (b) near field pattern showing hexagonal spot array.

peated, indicating a (nonlinearly modified) *Talbot-type effect* (Sturman et al., 1992), with contrast reversal occurring half-way between the Talbot imaging planes. The transverse period, calculated from the longitudinal period, is of the order of  $30\text{ }\mu\text{m}$ , indicating a far-field diffraction angle in agreement with our observed value.

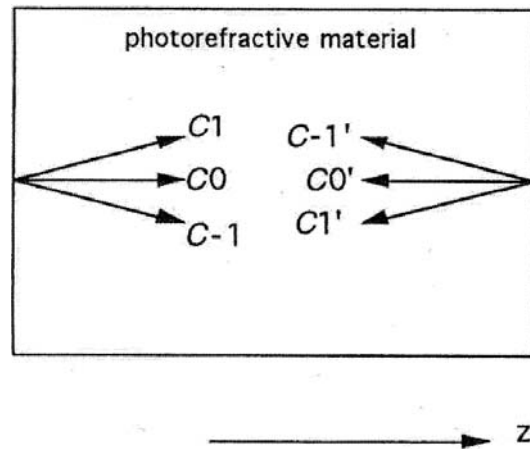
If the incident beam is slightly off-normal to the interface (typically by  $0.04^\circ$ ), and the power is increased, the entire hexagonal pattern rotates (Banerjee et al., 1995). The sense of rotation depends on the sense of the angular misalignment; thus, both clockwise and counterclockwise rotations of the pattern are possible through positive and negative angular misalignments. A typical value for the rotation speed in the steady state is  $100^\circ/\text{min}$  for an incident power of 30 mW.

In related experiments, Honda (1993) obtained similar results by using a  $\text{KNbO}_3$  crystal in index matching oil along with an external  $\text{BaTiO}_3$  self-pumped phase conjugate mirror. Also, a slightly converging beam was used by using a convex lens of focal length 300 mm. The reflectivity of the phase conjugate mirror was about 50%. Pattern formation has also been observed using a  $\text{KNbO}_3$  crystal along with a plane feedback mirror (Honda and Banerjee, 1996), similar to the arrangement discussed in Sec. 2. As will be discussed below, the angle of divergence depends on the length of the feedback path; thus it is possible to change the cone angle by adjusting the position of the external feedback mirror. In this case, the crystal *c*-axis should be slightly tilted from the beam axis to reduce the influence of the beam reflected from the back surface of the crystal. It has been also shown that pattern rotation can be achieved by using an additional erase beam, making a small angle with respect to the direction of propagation of the pump beams in the crystal (Honda, 1995; Uesu et al., 1998). The speed and sense of rotation of the hexagonal pattern in the far field may be also controlled with the erase beam. Hexagonal pattern formation has been observed in PR materials other than  $\text{KNbO}_3$ . Hexagonal pattern generation in Co-doped  $\text{BaTiO}_3$  with an external feedback mirror has been observed by Honda and Matsumoto (1995) and by Uesu et al. (1998). In all of the above experiments, higher-order hexagonal patterns have been observed in the far field with an increase of incident intensity. Also, interesting patterns have been observed using a single feedback system with a virtual feedback mirror. A virtual feedback mirror is achieved by inserting a lens between the exit plane of the crystal and the external feedback mirror (Honda et al., 1997). The lens images the mirror at a certain distance from the exit face of the crystal. Depending on the location of the lens, the image location could be outside or even inside the crystal. Square patterns have been observed using this arrangement. For an appropriate choice of the virtual feedback mirror (typically located inside the crystal), the hexagonal and square patterns have been shown to alternate

with time, demonstrating *pattern-hopping*, which is testimony to criterion #4 for self-organization in Sec. 1.

In a related experiment, *self-phase conjugation* similar to what was observed in SBN (Bogodaev et al., 1987) has been observed in  $\text{KNbO}_3$  (Kukhtarev et al., 1995). This configuration has applications in image processing as well, as recently shown by Banerjee et al. (1996). In a typical experimental setup involving  $\text{KNbO}_3:\text{Fe}$ , a wave  $C_1$  incident at about  $10^\circ$  to the normal to the crystal surface is reflected from the crystal, producing  $C_{-1}'$  (see Fig. 7). Due to scattering, additional waves  $C_0$  and  $C_0'$  develop, which propagate almost normal to the crystal surface. These represent concentric *Fresnel rings* and are analogous to Fabry-Perot modes in a resonator. With time, the inner ring may decompose into a hexagonal pattern. Furthermore, interaction of the four waves  $C_0$ ,  $C_0'$ ,  $C_1$ , and  $C_{-1}'$  gives rise to additional waves  $C_{-1}$  (counterpropagating to  $C_{-1}'$  and phase conjugate of  $C_1$ ) and  $C_1'$  (counterpropagating to  $C_1$  and its phase conjugate).

A variation of the above experiment involves interactions initiated by two beams  $C_1$  and  $C_0$ , and supported by reflections  $C_{-1}'$  and  $C_0'$ , to eventually produce  $C_{-1}$  and  $C_1'$ . In this case, we have found that if  $C_1$  represents the field from a point source, the phase conjugate  $C_{-1}$  images a certain distance behind the PR crystal. Furthermore, if  $C_1$  is the Fourier transform of an object, the phase conjugate of the object is recovered in the far field, traveling nominally in the direction of  $C_{-1}$ . By changing the position of the object with respect to the front focal plane of the Fourier transform lens, *edge enhancement* can be achieved (Banerjee et al., 1996).



**Figure 7** Six wave coupling in potassium niobate.

## 4 THEORY OF SELF-ORGANIZATION IN PHOTOREFRACTIVE MATERIALS

### 4.1 Fabry–Perot Modes

Assume that a radially symmetrical beam  $C(r)$ , where  $r$  represents the radial distance in the transverse plane, is nominally normally incident onto a *Fabry–Perot cavity* formed by the parallel faces of the PR material. The far-field intensity profile can be shown to be given by  $|\hat{C}(\theta)|^2 S(\theta)$ , where  $S(\theta)$  is a shaping function which, to a first approximation, can be shown to be

$$S(\theta) \propto \frac{1}{1 + F(\theta) \sin^2 k_0 \theta^2 L/2}. \quad (4-1)$$

Also,  $\hat{C}(\theta)$  represents the Fourier transform of  $C(r)$ , with  $\theta = k_r/k_0$ , where  $k_r$  is the spatial corresponding to  $r$  and  $k_0$  is the propagation constant of the light in the medium. In the above relation,  $F$  is the cavity *fineness* and  $L$  is the thickness of the material. From Eq. (4.1), upon setting  $k_0 \theta^2 L/2 = \pi$ , the semi-angle of the first ring can be calculated to be approximately  $0.4^\circ$  in the material, which is of the order of our observed value of  $0.8^\circ$  in air. With time, the ring may break up into hexagonal spots, as observed experimentally. Note also that in the experiments, secondary (or higher-order diffraction) rings, and sometimes higher-order hexagonal spots, are also observed. We have observed in our experiments that the radius of the second ring is observed to be  $\sqrt{3}$  times that of the first, which can be also derived from Eq. (4.1) by setting  $k_0 \theta^2 L/2 = 3\pi$ .

We would like to point out that the existence of Fabry–Perot modes in the crystal cavity supports the concept of periodic imaging during propagation in the crystal. In the so-called “open” cavities consisting of a matched or misaligned PR crystal and an external feedback mirror, the concept of Talbot imaging has been used to determine the scattering angle (Banerjee et al., 2000). However, we feel that the concept of Talbot imaging as discussed in Honda and Matsumoto (1995) can only be applied to the case of propagation in the “cavity” between a “thin” slice of the PR material containing the induced reflection grating and the external mirror. More on this is discussed later.

In a nonlinear system where the incident beam may originate from light scattering, the coupling between forward and backward traveling waves may be provided by transmission and/or reflection gratings. From experimental results on beam coupling, it has been shown that reflection gratings are dominant. In the remainder of this chapter, we will assume only reflection gratings to be present. It turns out that the *scattering angle* will be nonlinearly modified depending on the strength of the reflection grating.

## 4.2 Model Equations

We represent the forward and backward traveling waves in the nonlinear PR material as

$$E = \text{Re}[(C_e \exp -jk_0 z + C_{e'} \exp jk_0 z) \exp j\omega_0 t] \quad (4-2)$$

where  $C_{e,e'}$  denote the forward and backward traveling wave amplitudes, respectively, and  $\omega_0$  is the angular frequency of the light in the medium. We also assume that the material has light-induced changes in the refractive index due to reflection gratings formed in the material, with spatial frequency  $2k_0$ . The spatial evolution of the forward and backward traveling envelopes can then be written as

$$L_e C_e = -jk_0 \delta n C_{e'}, \quad L_{e'} C_{e'} = -jk_0 \delta n^* C_e \quad (4-3)$$

where  $L_{e,e'}$  are linear operators given by the relation

$$L_{e,e'} = \partial / \partial z \mp j(1/2k_0) \nabla_{\perp}^2. \quad (4-4)$$

The variable  $\delta n$  represents the fractional change of refractive index due to the *induced reflection grating* and evolves according to

$$\tau \partial \delta n / \partial t + \delta n = \gamma \frac{C_e C_{e'}^*}{|C_e|^2 + |C_{e'}|^2}. \quad (4-5)$$

Equations (4-3) and (4-5) are similar to Eqs. (2-3) and (2-4) derived in connection to the Kerr media in Sec. 2.

We consider reflection gratings only for now because they are dominant in PR potassium niobate (Honda, 1993). Transmission gratings have been assumed in other analyses, such as for the determination of the onset of instability (Saffman et al., 1993). By using the model of excitation of satellite beams due to propagation of contrapropagating primary or pump beams, and transmission gratings, the so called *spatial dispersion curves* for the onset of instabilities leading to satellite beam formation have been derived. The plots show the dependence of the minimum threshold gain as a function of the angle between the pump and the spatial sidebands (Saffman et al., 1993). Dispersion curves assuming predominantly transmission gratings and aided by reflection gratings have been also derived by Kukhtarev et al. (1995). Later on in the chapter, we will provide the results of such dispersion curves but using reflection gratings in the model, since it pertains more closely to spontaneous pattern generation in PR potassium niobate.

We would like to point out that *pattern dynamics* has been extensively studied in a bidirectional PR *ring resonator* assuming transmission grating

approximation and four-wave mixing in the active PR medium (Chen and Abraham, 1995). Spontaneous *symmetry breaking*, *dynamical oscillations*, *vortex formation*, and complex pattern development are predicted for large Fresnel numbers. A PR oscillator with a stable resonator has been used to model a nonlinear dynamical system in which *transverse mode patterns* have been observed (Malos et al., 1996).

## 5 INSTABILITY CRITERION AND THE DISPERSION RELATION

There is considerable work done on the onset of instabilities in a PR medium with reflection gratings due to counterpropagation of pump beams. The analysis of Sturman and Chernykh (1995) assumes a medium in which there is no energy coupling. Saffman et al. (1994) has performed a more detailed analysis assuming both real and complex coupling coefficients. Honda and Banerjee (1996) have improved on their analysis by showing that pattern generation can occur even for purely energy coupling.

We now present the *threshold condition* for instabilities derived for the experimental arrangement in Honda and Banerjee (1996) with the PR crystal and a feedback mirror (similar to the setup in Fig. 1). We use relations (4.3)–(4.5) and substitute

$$\begin{aligned} C_{e,e'} &= C_{0,0'} [1 + c_{1,i'} \exp -j\mathbf{K} \cdot \mathbf{r} + c_{-1,-i'} \exp j\mathbf{K} \cdot \mathbf{r}], c_{i,i'} \\ &= C_{i,i'} / C_{0,0'} \end{aligned} \quad (5-1)$$

where  $K = |\mathbf{K}|$  is the transverse wave number and  $r$  denotes the transverse coordinate to get

$$\begin{aligned} (\partial/\partial z - jk_d)c_1 &= jA\gamma(c_1 + c_{-1}^* - c_{1'} - c_{-1'}), \\ (\partial/\partial z + jk_d)c_{-1}^* &= -jA\gamma^*(c_1 + c_{-1}^* - c_{1'} - c_{-1'}), \\ (\partial/\partial z + jk_d)c_{1'} &= jA\gamma^*(c_1 + c_{-1}^* - c_{1'} - c_{-1'}), \\ (\partial/\partial z - jk_d)c_{-1'} &= -jA\gamma(c_1 + c_{-1}^* - c_{1'} - c_{-1'}), \end{aligned} \quad (5-2)$$

where  $k_d = K^2/2k_0$  and  $A = A(z) = |C_0|^2|C_{0'}|^2/[|C_0|^2 + |C_{0'}|^2]$ . Note that since  $A$  is a function of  $z$ , Eq. (5-2) cannot be solved analytically. However, when the reflectivity of the feedback mirror is unity or the reflection from the back surface of the crystal is considerable, we can approximate  $A$  by 1/4 (Honda and Banerjee, 1996).

For the case of a feedback mirror placed behind the crystal (similar to the schematic in Fig. 1, with the Kerr medium replaced by the PR crystal),

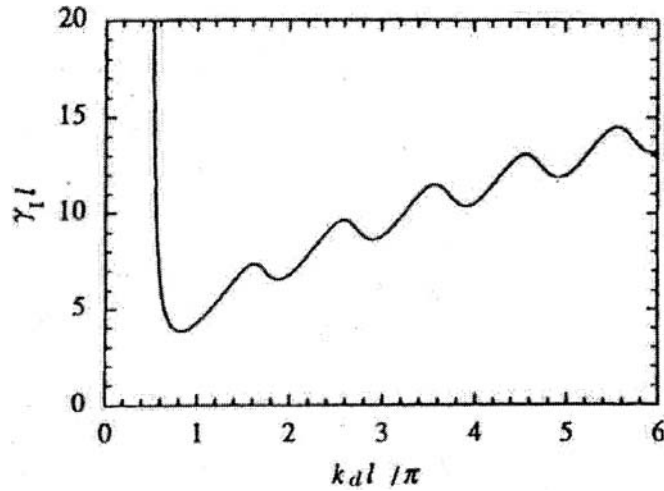
the boundary conditions can be written as

$$\begin{aligned} c_{1,-1}(0) &= 0, \\ c_{1'}(L) &= \exp(-2jk_d l)c_1(L), \quad c_{-1'}(L) = \exp(2jk_d l)c_{-1}^*(L) \end{aligned} \quad (5-3)$$

where  $L$  is the crystal thickness and  $l$  denotes the distance between the PR medium and the feedback mirror. The threshold condition can be found using Eqs. (5-2) and (5-3) and using the Laplace transformation to solve. Assuming that the mirror is placed against the back surface of the crystal, the *dispersion relation* can be written as

$$\cos wL \cos k_d L + (\gamma/2w) \sin wL \cos k_d L + (k_d/w) \sin wL \sin k_d L = 0, \quad (5-4)$$

where we have assumed the *coupling constant*  $\gamma$  to be purely imaginary ( $\gamma \rightarrow -j\gamma$ ) and  $w^2 = k_d^2 - \gamma^2/4$ . Fig. 8 shows the dispersion curve for this case. When  $\gamma$  is just above the threshold for spatial sideband generation, the direction of the sidebands will correspond to  $k_d$  which gives the minimum of the dispersion curve. For other mirror locations, the angle between the carrier and the spatial sidebands decreases as shown in Honda and Banerjee (1996).



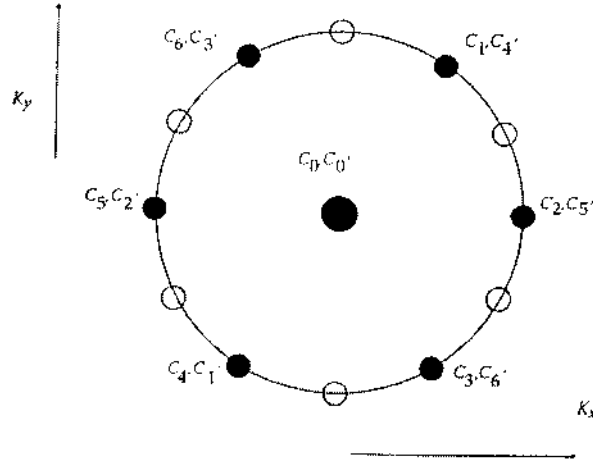
**Figure 8** Threshold condition assuming purely energy coupling and for mirror feedback. Mirror is located at the back surface of the sample.

## 6 NONLINEAR EIGENMODES IN THE STEADY STATE

The formulation stated above through Eqs. (4-3)–(4-5) can also be used to study the exact spatial behavior of the carriers (contrapropagating pumps) and the spatial sidebands. In the steady state, the spatial evolution of the carriers and the spatial sidebands lying on a scattering ring can be studied by solving the system of equations (Dimmock et al., 2000)

$$\begin{aligned}
 L_i C_i &= \sum_{j'} \delta n_{ij'} C_{j'} = \gamma \sum_{j'kl'} C_k C_{l'}^* C_{j'} / \left[ \left| \sum_i C_i \right|^2 + \left| \sum_i C_{i'} \right|^2 \right] \\
 L_{i'} C_{i'} &= \sum_j \delta n_{ji'}^* C_j = \gamma^* \sum_{j'kl'} C_{k'} C_{l'}^* C_j / \left[ \left| \sum_i C_i \right|^2 + \left| \sum_i C_{i'} \right|^2 \right]
 \end{aligned}
 \tag{6-1}$$

where we have assumed the optical properties of the PR material to be isotropic. As seen from Eq. (6-1), coupling will occur only between waves whose transverse wavevectors satisfy the general relation  $\mathbf{K}_i + \mathbf{K}_{i'} = \mathbf{K}_k + \mathbf{K}_{k'}$ . An example of a set of contrapropagating pumps and a set of six forward and backward propagating scattered sidebands is shown on the transverse  $\mathbf{K}$ -plane in Fig. 9. It can be shown that there can be seven different types of

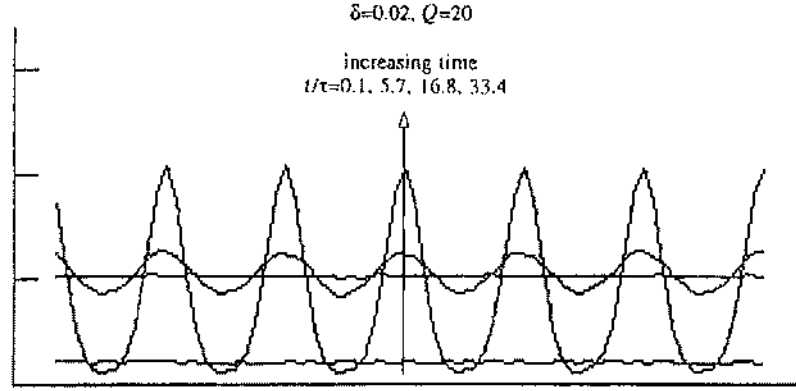


**Figure 9** Transverse  $\mathbf{K}$ -vectors of hexagonally related scattered waves w.r.t. that of the forward and backward propagating beams. Two sets of hexagonally related scattered beams are shown.

couplings that may occur. For example,  $\mathbf{K}_l = 0$ ,  $\mathbf{K}_i = \mathbf{K}_k + \mathbf{K}_j$  couples the main forward traveling beam with three waves that are hexagonally related. The interaction  $\mathbf{K}_i = -\mathbf{K}_l$ ,  $\mathbf{K}_k = -\mathbf{K}_j$  couples sets of hexagonally related waves together.

In what follows, we assume a geometry identical to the experimental arrangement in Banerjee et al. (1995) with only the unmatched crystal and no feedback mirror. Using Eq. (6-1) as a model, the spatial evolution of the carriers and the sidebands has been analyzed for the case when there are 72 sidebands symmetrically distributed on the scattering ring (Dimmock et al., 2000). The preliminary results which were performed using a purely imaginary  $\gamma$  show the general nature of the *modes* in the steady state that can exist within the interaction region in the PR material. These modes show the permissible values of the phase difference between the pump and the sidebands at the front surface of the material for different values of the gain parameter proportional to  $\gamma$ . Furthermore, one can simultaneously get the spatial variations of the contrapropagating pumps and the spatial sidebands, assumed equal in magnitude for simplicity. The results therefore define the conditions needed for self-organization of the laser beam into a scattering ring, starting from *fanning noise* in the material. However, it turns out that the ratios of scattered to pump intensities both in the forward and the backward directions are not exactly similar to experimentally observed results (Banerjee et al., 1995).

We would like to point out that a simple time evolution simulation to illustrate the basic principle of the formation of the scattering ring and hexagon formation can be performed by starting from models (4-3)–(4-5) and even assuming an (intensity-independent) imaginary coupling constant, constant amplitudes for the interacting pumps, and spatial sidebands in the PR material, and assuming a thin sample (Dimmock et al., 2000). Taking an initial linear scattering from beam fanning, the evolution of the spatial sidebands into a scattering cone and eventually into hexagonal pattern in the far field is shown in Fig. 10. The plots show that the energy scattered into the ring as the first stage of the self-organization process essentially later redistributes into the hexagons. The plots are quantitatively modified slightly if transmission gratings are also incorporated into the simulations. All simulation results are in qualitative agreement with experimental observations (Banerjee et al., 1995). If one monitors the minimum value of the gain needed for the onset of instabilities as a function of the linear scattering parameter, it is observed that the threshold gain decreases sharply with increasing initial linear scattering, as expected, and tends to slightly decrease for very high values of the scattering parameter. This increase can be attributed to the fact that excessive linear scattering tends to deplete the pumps of their initial energy, thus inhibiting the formation of reflection



**Figure 10** Time evolution of scattering around a circle, showing the growth of the scattering ring and eventual formation of the hexagonal spot pattern. The normalized coupling parameter  $Q$  showing coupling between hexagonally related points on scattering ring and the pump is taken to be 20, and the linear scattering coefficient  $\delta$  from the pump (to initiate the self-organization process) is taken to be 0.02.

gratings and eventual transfer of pump energies into the spatial sidebands. This simple simulation also demonstrates the justification for looking for exact spatial eigenmodes which depict the spatial variation of the pumps and the scattering ring as the first stage of the self-organization process.

As stated before, the discrepancy between numerical simulations for the nonlinear eigenmodes and experimental finding of the energy scattered into the ring and eventually into the hexagon can be resolved by assuming a complex coupling constant. Possible reasons for the nonideal phase of the coupling coefficient are as follows. As in any PR material, the contribution to photorefractivity can come from both *diffusion* and *photovoltaic* contributions. While diffusion creates a space charge field which is out of phase with the intensity profile, photovoltaic effects give rise to space charge fields which are in phase with the intensity (Yeh, 1993). In general, therefore, an arbitrary phase difference may exist. Furthermore, even for a purely photovoltaic material, it has been shown that there can exist a phase difference between the intensity grating and the fundamental spatial frequency component of the space charge field for large modulation depths. This can also give rise to a complex coupling coefficient (Kukhtarev et al., 1998).

Starting from Eq. (6-1) and setting  $\gamma \rightarrow \gamma \exp(j\phi)$  with

$$C_{i,j} = S_{i,j}(z) \exp(-j\mathbf{K}_{i,j} \cdot \mathbf{r}) \exp(\mp j\pi z/L) \exp(j\phi_{i,j}(z)) \quad (6-2)$$

we can derive the spatial evolution equations for the amplitudes and phases of the interacting waves. If we assume that the amplitudes and phases  $S_{i,i'}$ ,  $\phi_{i,i'}$ ,  $i, i' \neq 0$ , of the interacting waves on the scattering ring are identical for simplicity, we get, after extensive algebra, coupled differential equations which have the functional forms (Dimmock et al., 2000)

$$\begin{aligned}\partial S_0^2 / \partial z &= (\gamma/I) F_0 [S_0^2, S_1^2, S_0^2, S_1^2, b, c, \phi, N] - N \delta S_0^2, \\ \partial S_1^2 / \partial z &= (\gamma/I) F_1 [S_0^2, S_1^2, S_0^2, S_1^2, b, c, \phi, N] + \delta S_0^2, \\ \partial b / \partial z &= (\gamma/I) F_b [S_0^2, S_1^2, S_0^2, S_1^2, b, c, \phi, N] + \pi/L,\end{aligned}\tag{6-3}$$

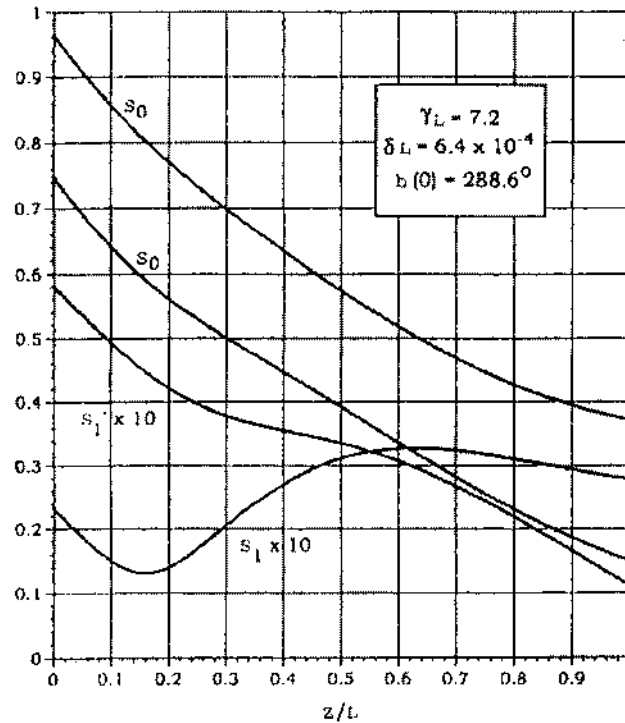
where

$$b = \phi_{1'} - \phi_0 + \pi z/L, \quad c = \phi_1 - \phi_0 - \pi z/L.\tag{6-4}$$

The corresponding equations for  $S_{0',1'}^2$ ,  $c$  can be found by interchanging the primed and unprimed variables, interchanging  $b$  and  $c$  in the above equations, and replacing  $L$  by  $-L$ . In Eq. (6-3),  $I$  is the incoherent intensity and we should point out that Eq. (12) is valid assuming up to third-order interactions. The constant is a linear scattering parameter.  $N$  is the number of interacting waves on the scattering ring, taken here to be equal to 72. The exact expressions for  $F_i$  in Eq. (6-3) are given in Dimmock et al. (2000). Conservation rules for waves interacting through the formation of reflection gratings hold, and Eq. (6-3) is solved numerically assuming boundary conditions pertinent to the front and back surfaces of the crystal which generate the counterpropagating waves in a truly mirrorless configuration.

The numerical results (not shown here) show the existence of multiple eigenmodes which are possible in the PR medium. Each *eigenmode* is characterized by a value of  $b(0)$  [assumed equal to  $c(0)$ ] and is the locus of permissible solutions on the  $b(0)$ -plane. If the forward and backward *scattering ratios*, defined as the fraction of the pump energy scattered onto the ring, are monitored, it follows that by relaxing the condition on  $\phi$ , viz., making it arbitrary, it is possible to attain values similar to experimental observations. For instance, for a value of  $\phi = 230^\circ$ , about 40% of the energy is scattered into the ring in the forward direction, with about 30% in the backward direction, in close agreement with experimental observations (Banerjee et al., 1995). The fact that a *complex coupling constant* is required to achieve the expected forward and backward scattered energies corroborates the fact that the ideal phase difference (viz.,  $270^\circ$ ) between the intensity grating and the induced refractive index profile is probably changed due to

contributions from the diffusion contribution to the PR effect and from the finite modulation depth of the intensity grating, as explained earlier. The phase difference between the pump and the sidebands at the front surface is close to  $270^\circ$  for this case, which implies phase modulation of the profile of the total beam at this plane (and also at the exit plane), with amplitude modulation in the center of the PR material. The analysis also enables us to track the exact spatial evolution of the pumps and the spatial sidebands; this is shown in Fig. 11. We would also like to point out that the agreement between theory and experiment is only observed for the above value of  $\phi$ , which explains why self-organization is not observed when the experiment is performed with the  $c$ -axis of the crystal turned in the reverse direction (Banerjee et al., 1995).



**Figure 11** Spatial variation of the forward and backward scattered and main beam amplitudes during propagation through the crystal.

## 7 MODEL OF HEXAGONAL FORMATION BASED ON TRANSVERSE ELECTRICAL INSTABILITY

In what follows, we will discuss the contribution of *electrical instabilities* to the formation of hexagonal structures. Adequate description of self-organized pattern in  $\text{KNbO}_3$  and other thick PR materials include material equations (like diffusion-drift model) and the Maxwell wave equation. Both material and optical equations are nonlinear and potentially are capable to describe the formation of spatial-temporal patterns. As an example, we can mention the problem of *holographic subharmonic* (Mallick et al., 1998), observed during self-diffraction of two beams with slightly different frequencies in the external electrical field in  $\text{Bi}_{12}\text{SiO}_{20}$  crystals. Appearance of additional beam between intersecting “pump beams” was originally explained by optical nonlinearities (Jones and Solymar, 1989). Later, it was realized that instabilities of material equations, like *period doubling*, lead to the formation of subharmonic component in the space-charge field and in the refractive index (Sturman et al., 1992). Similar trends are visible in the explanation of hexagon patterns in the thick PR materials.

All previous explanations of pattern formation in PR crystals were based on instabilities of optical equations (Maxwell wave equations) where material equations play only insignificant role. Only recently it was realized that photogalvanic currents may be responsible for *contrast enhancement* and may result in the *space-charge instabilities* (Kukhtarev et al., 1994). Quantitatively, formation of the spatial patterns due to photogalvanic current may be explained taking into account the relation  $\nabla \cdot \mathbf{J} = 0$ . This equation implies that the current has a vortex structure and forms closed loops. Detailed calculations of the transversal structure caused by photogalvanic nonlinearity are beyond the scope of this discussion. Suggesting that *photogalvanic instabilities* lead to transverse pattern of the  $E$ -field and refractive index, let us discuss experimental results in near field.

Transversal modulation of the refractive index of a thick PR crystal may be regarded as a recording of bunch of *optical channels* or *waveguides*. As it was shown in Kukhtarev et al. (1994), modulation of the refractive index may be visualized in the near field as optical channeling. We can thus model transversal modulation of the dielectric constant by the function

$$\varepsilon(x, y) = \varepsilon_0 + \varepsilon_x \cos K_x x + \varepsilon_y \cos(K_y y + \varphi) \quad (7-1)$$

where  $\varepsilon_0$  denotes the average value of the permittivity and  $\varepsilon_{x,y}$  are the amplitudes of the modulation along the transverse  $x$ - and  $y$ -axis, with wave numbers  $K_x$ ,  $K_y$ , and proper phase shift  $\varphi$ . Introducing the function  $\varepsilon(x, y)$  in

Maxwell's equations, we can get the following result for the near-field intensity:

$$I(x, y) = I_0[1 + \varepsilon_x(L_x/\lambda)^2 \sin^2(\pi\lambda z/2L_x^2)\cos K_x x + \varepsilon_y(L_y/\lambda)^2 \sin^2(\pi\lambda z/2L_y^2)\cos(K_y y + \varphi)] \quad (7-2)$$

where  $L_{x,y} = 2\pi/K_{x,y}$  and  $\lambda$  is the wavelength. The solution (7-2) is valid for small modulation and includes longitudinal modulation with the periods

$$z_{x,y} = 2L_{x,y}^2/\lambda. \quad (7-3)$$

We can see that Eq. (7-3) also describes contrast inversion. As described earlier, for experimental values with  $\text{KNbO}_3$  ( $\lambda = 0.514 \mu\text{m}$ ,  $L_x = L_y = 30 \mu\text{m}$ ), we can get for longitudinal period  $z_x = z_y = 0.49 \text{ cm}$  that is very close to experimental value of  $0.5 \text{ cm}$  (Banerjee et al., 1995).

It is important to note that the explanation of hexagonal structure by Talbot effect imaging is valid only for optically thin gratings, where Talbot effect description during the propagation in free-space is justified. In our case, we have used a thick crystal (1 cm thick), and we should use the adequate model of thick holographic gratings. The channeling effect is pronounced for thick gratings and naturally describes the effect of contrast inversion. In contrast to Bragg diffraction that normally needs coherent light, channeling may be observed also in incoherent illumination.

## 8 POTENTIAL APPLICATIONS

We have summarized, using minimal mathematics, some important aspects of an area which is rather complicated for two reasons: (a) because of the nonlinear and spatio-temporal nature of the problem and (b) because the response of a nonlinear material to incident light is a complicated phenomenon, governed by a set of nonlinear coupled differential equations. Wherever possible, experimental results have been quoted or referred to in order to assure readers that there is some connection to reality behind the complicated mathematics. At this time, one can contemplate on potential applications of the self-organization process. It is hoped that a detailed understanding of the self-organization process will enable researchers to think of even more applications in real life. Some of the possibilities for applications, using PRs as the nonlinear material, are listed below (Banerjee et al., 2000).

- (1) We anticipate that the self-organization can readily be used to intelligently manufacture *diffractive optical elements*, such as hex-

agonal arrays, gratings, etc. In this case, one can use the nonlinear properties of the active material to create diffractive optical elements, rather than rely on complicated geometrical processing. In the long run, these patterns can be generated and stored in thin-film PR polymers. In the shorter term, one can image a plane inside of a thick crystal on a film and thereby make such diffractive elements.

- (2) The near-field pattern is observed to be comprised of hundreds of phase-related spots in a hexagonal array which can be caused to shift or move across the face of the crystal. We believe that this can, in principle, be used for *hexagonal sampling of images* in digital image processing which offers spatial bandwidth savings (Mersereau, 1979). Hexagonal array generation has been traditionally done by fabricating *binary phase gratings* (Roberts et al., 1992). These hexagonal arrays can also be used to effectively couple light into a fiber bundle, which may eventually feed into adaptive antenna array structures.
- (3) The far-field pattern can be used to *broadcast* separate *images* of an input pattern in different directions. Furthermore, because these separate images have specific phase relations, unique image processing can be performed by interfering these separate images with each other or with the original beam. It is also conceivable that the far-field pattern comprising six peripheral spots and the central spot can be used to monitor velocity and acceleration of a moving body.
- (4) As both near- and far-field pattern rotations are extremely sensitive to small misalignments of the pump beam with respect to the crystal surfaces and axes (Banerjee et al., 1995; Kukhtarev et al., 1995), we anticipate that this material can be employed as an integral element in *misalignment detection* or rotation sensing devices.
- (5) The *self-phase conjugation* can be used to form conjugate images in both forward and backward directions without the need of complex additional optics. *Edge enhancement*, an important aspect of image processing, has been demonstrated using this material, and there is potential to develop real-time optically *edge-enhanced correlators* using this concept.
- (6) Finally, since we can observe and measure holographic currents during grating recording in the PR material (Noginova et al., 1997), we anticipate that the self-organization effects and their time dependencies can be modified and indeed controlled by application of external electrical fields to the KNbO<sub>3</sub> crystal. In the long term, the possibility of superposing external electrical mod-

ulation to change the holographic current and hence the diffraction pattern in the near and far fields seems feasible. This will open the door to exciting applications of the crystal in nonlinear information and image processing which may be electronically controllable.

## 9 PROBLEMS

1. Assume that one arm of a unidirectional ring cavity such as shown in Fig. 1 in Chap. 5 has a Kerr-type nonlinear material with a nonlinear refractive index coefficient  $n_2$  while the feedback has a low-pass spatial filter that can pass all spatial frequencies from zero (d.c.) to a certain maximum frequency (cutoff frequency)  $K_0$ . Determine and plot the transmission characteristic as a function of the spatial frequency for (a) a focusing medium and (b) a defocusing medium.
2. Use the beam propagation method to model the evolution of the complex amplitudes of contrapropagating optical waves in a nonlinear cavity as given by Eq. (2-3) where the refractive index is given by Eq. (2-4). Assume steady state for the refractive index. Use any nonzero initial values for the counterpropagating optical wave envelopes.
3. In the case of self-organization leading to hexagon formation, show that secondary hexagons may form on a circle whose radius is  $\sqrt{3}$  times the radius of the circle on which the primary hexagon forms. Show physically (a vector diagram will help) how the hologram vectors responsible for producing the primary hexagon spots can contribute to the formation of the secondary hexagons.
4. Find the so-called dispersion relation describing the onset of instabilities in the case when the gratings are transmission instead of reflection. Use the experimental arrangement of a thin PR medium and a feedback mirror as a model for your calculations.
5. Extend the calculations for the dispersion relation for the case when the gratings are predominantly reflection, but there are weak transmission gratings as well. Compare your results with the dispersion relations obtained from pure transmission and pure reflection gratings.
6. How is the near-field hexagonal array pattern related to the far-field hexagonal spots? In the case when the hexagonal spots rotate in the far field, what happens to the near-field picture?

Suppose that the far-field pattern comprises just a central spot with diametrically opposite satellite spots. What is the corresponding near-field picture? Give reasons for your answer.

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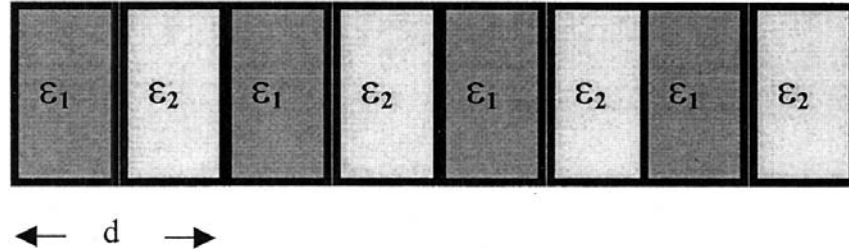
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## Nonlinear Optics of Photonic Bandgap Structures

### 1 INTRODUCTION

Thus far in this book, we have examined nonlinear optical propagation in a wide variety of structures and media. We have examined forward and backward propagation of light due to induced reflection gratings in nonlinear materials, such as photorefractive (PR) crystals. In particular, we have seen in Chap. 9 how two-wave mixing is possible because of induced reflection gratings in PR materials. Also, in Chap. 11, we have seen how multiwave forward and backward mixing in PR materials can give rise to self-organized patterns such as hexagons. The key point in both cases is that the reflection grating is induced; that is, the grating period is exactly one half of the wavelength of the contrapropagating light waves in the medium. Another way of stating this is to say that the grating vector length is twice the propagation constant of the light in the medium. In this case, there is perfect phase matching between the forward and the backward traveling waves, which gives rise to substantial reflected light.

Now consider the following scenario: Assume that the grating is built into the material a priori; that is, it is not induced because of the material response. In this case, one can have the liberty of making gratings of any arbitrary period or wave vector in the material (see Fig. 1). Hence if the fundamental spatial frequency of the grating is exactly equal to twice the propagation constant of the contrapropagating light waves, one can expect maximum reflection of an incident light wave. However, if this is not the case, the interaction is no longer phase matched and the reflection is not as high. In other words, given a certain grating period, there is a band of frequencies for which appreciable amount of light is reflected back, leading to minimal transmission. This implies that if the frequency of light is changed, there may be a certain stop band, or *bandgap*, in the transmission. Such periodic



**Figure 1** Schematic diagram of a one-dimensional periodic layered structure. Propagation is along the horizontal direction ( $\pm z$ ).

structures are referred to as *photonic crystals* or *photonic bandgap structures*. In this chapter, we will restrict ourselves to one-dimensional photonic bandgap structures. In general, they can exist in higher dimensions as well, but this is beyond the scope of this book. Interested readers are referred to Johnson and Joannopoulos (2002).

In this chapter, we will first derive the linear dispersion relation for propagating optical waves through a one-dimensional photonic bandgap structure. It turns out that the dispersion relation is similar to that of a crystal lattice, and possesses stop bands in the transmission. In some cases, dispersion around the stopband can be approximated as a wave-guide-type dispersion. Thereafter, we will examine the effect of a cubic nonlinearity on wave propagation through a one-dimensional photonic bandgap structure. Because of the balance between the nonlinearity and the dispersion, one can sometimes expect solitons in the structure. These solitons are called *gap solitons*. Continuous wave (CW) propagation through these structures under the action of nonlinearity and dispersion can give rise to *hysteresis* and *bistability* in the transmission. Finally, we discuss *second harmonic generation* in photonic bandgap structures with quadratic nonlinearity. It turns out that one can enhance second harmonic generation through phase matching using the dispersion in these structures. We treat the simple case of an undepleted pump or fundamental, and show enhancement of the second harmonic from the structure.

## 2 THE LINEAR DISPERSION RELATION

As seen in Chap. 1, the *dispersion relation* for a system can be found by writing down the pertinent wave equation modeling the propagation and substituting the expression for a propagating wave of the type  $\exp j(\omega t - kz)$

to find the resulting relation between the frequency  $\omega$  and the propagation constant  $k$ . We extend this concept to the case of contradirectional propagation of plane waves, as is the case for a photonic bandgap structure. We first start from the wave equation for the optical field  $E$  of the form (Martijn de Sterke and Sipe, 1994)

$$\frac{\partial^2 E}{\partial z^2} - \frac{\epsilon_r(z)}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \quad (2-1)$$

where

$$\epsilon_r(z) = \epsilon(z)/\epsilon_0 = n_0^2 + \Delta\epsilon(z) \quad (2-2)$$

denotes the *permittivity modulation* along  $z$ , and  $n_0$  denotes the average refractive index of the material. Depending on the nature of the periodic refractive index modulation, one can suitably express the permittivity modulation  $\Delta\epsilon(z)$  in terms of a Fourier series of the form

$$\Delta\epsilon(z) = \sum_{m \neq 0} \epsilon_m \exp -jmK_g z \quad (2-3)$$

with the grating wave number  $K_g = 2\pi/d$ , where  $d$  is the spatial period of the longitudinal permittivity grating. For simplicity, we will examine the case where the permittivity variation is sinusoidal, i.e.,  $\epsilon_1 = \epsilon_{-1}$  and all other  $\epsilon_m$ 's are equal to zero. In this case, Eq. (2-3) reduces to

$$\Delta\epsilon(z) = 2\epsilon_1 \cos K_g z. \quad (2-4)$$

In actuality, optical waves with a wavelength equal to twice the spatial period of the grating will be strongly reflected back. This is because light which is *Fresnel reflected* from every period of the grating will add up in phase, leading to enhanced reflection. With this in mind, we will take the propagation constant of the light to be  $k_0 = \pi/d$  and set  $K_g = 2k_0$  in Eqs. (2-3) and (2-4), to derive the dispersion relation around  $(\omega_0, k_0)$ .

Now the optical field should be expressed as a sum of forward and backward traeling waves of the form

$$E(z, t) = (1/2)[E_{e+}(z, t)\exp j(\omega_0 t - k_0 z) + E_{e-}(z, t)\exp j(\omega_0 t + k_0 z) + c.c.] \quad (2-5)$$

where  $\omega_0 = k_0 c/n_0$ , and  $E_{e+}, E_{e-}$  denote the slowly varying forward and backward traveling envelopes, respectively. Note that the expressions for the optical field and the permittivity profile is consistent with our discussion in Sec. 1 that the grating vector length is nominally twice the propagation constant of the light in the medium. Now upon substituting Eqs. (2-4) and (2-5) into Eq. (2-1) and separating the coefficients of  $\exp [j(\omega_0 t \mp k_0 z)]$ , we can obtain a pair of *coupled mode equations* describing the spatiotemporal evolution

of the forward and backward traveling envelopes  $E_{e+}, E_{e-}$  (Martijn de Sterke and Sipe, 1994):

$$\begin{aligned} \partial E_{e+}/\partial z + (n_0/c)\partial E_{e+}/\partial t + j\kappa E_{e-} &= 0 \\ -\partial E_{e-}/\partial z + (n_0/c)\partial E_{e-}/\partial t + j\kappa E_{e+} &= 0 \end{aligned} \quad (2-6)$$

where

$$\kappa = \omega_0 \varepsilon_1 / 2n_0 c. \quad (2-7)$$

In deriving Eq. (2-6), the slowly varying envelope approximation has been used to neglect the second derivatives of  $E_{e+}, E_{e-}$  w.r.t.  $t$  and  $z$ .

To now obtain the dispersion relation for the propagation of the envelopes, we need to assume variations of  $E_{e+}, E_{e-}$  of the form

$$E_{e\pm} = A_{\pm} \exp j(\Omega t - Kz), \quad (2-8)$$

where  $\Omega = \omega - \omega_0$ ,  $K = k - k_0$  denote the sidebands around the carrier. Substitution of this assumed form for  $E_{e+}, E_{e-}$  into the coupled mode equations yields the matrix equation

$$\begin{bmatrix} n_0 \Omega / c - K & \kappa \\ \kappa & n_0 \Omega / c + K \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = 0. \quad (2-9)$$

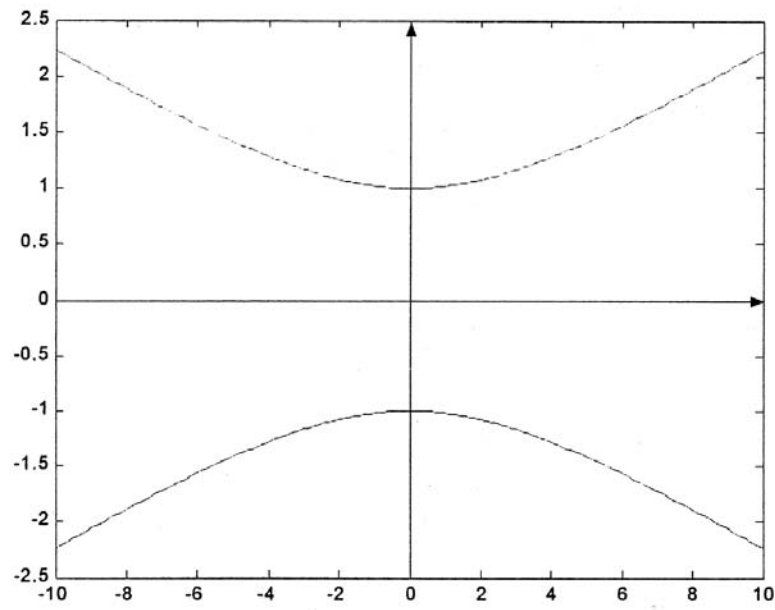
The above system has two independent solutions for  $A_+, A_-$  if the determinant of the matrix is zero. This yields the *dispersion relation*

$$\Omega^2 = (c/n_0)^2 (K^2 + \kappa^2) = (c/n_0)^2 K^2 + \Omega_c^2; \quad \Omega_c = c\kappa/n_0. \quad (2-10)$$

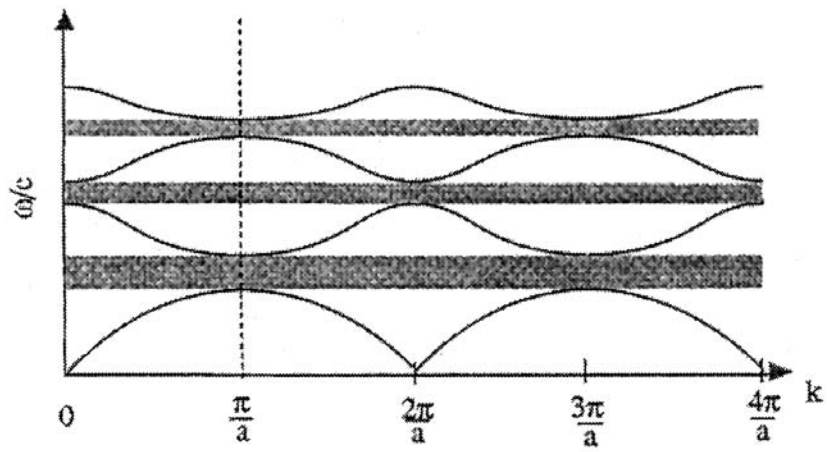
This is plotted in Fig. 2a.

Note that the dispersion relation is similar to the dispersion relation for a rectangular waveguide. Also, observe that there is no propagation below the cutoff frequency  $\pm \Omega_c$ , implying a *photonic stop gap* or bandgap of  $\Delta\Omega = \Delta\omega = 2\Omega_c = 2c\kappa/n_0$  located about  $\omega_0, k_0$ . Using the definition of  $\kappa$  in Eq. (2-7), the bandgap can be reexpressed as  $\Delta\omega/\omega_0 \approx \Delta n/n_0$ , where  $\Delta n$  represents the peak change in the refractive index. In other words, not only is the frequency  $\omega_0 = k_0 c/n_0 = \pi c/n_0 d$  Bragg reflected, there exists a band of frequencies  $\Delta\omega$  around it that also experience appreciable reflection, and hence do not propagate. This is in agreement with the heuristic picture of enhanced Bragg diffraction within a range of frequencies (detuning), as is common to any grating problem.

A more detailed analysis of propagation through a one-dimensional periodic layered structure yields a more exact dispersion relation as shown in Fig. 2b. This can be rigorously derived using the concept of the *Floquet–Bloch functions* and the *Kronig–Penney model*, and is outside the scope of this book. Interested readers are referred to Haus (1996).



(a)

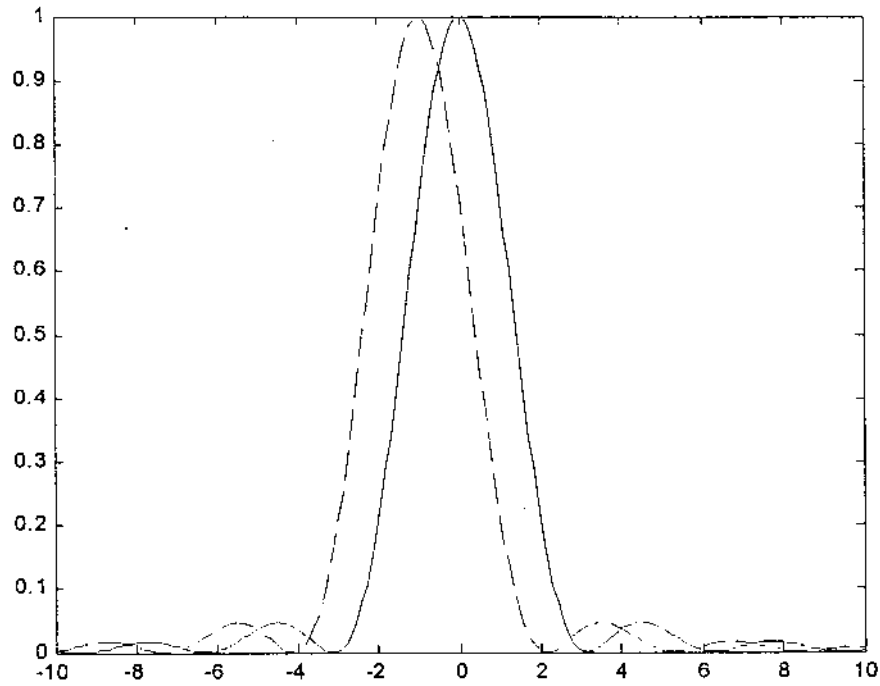


(b)

**Figure 2** (a) Dispersion relation as derived from Eq. (2-10). (b) More exact dispersion relation (Haus, 1996). The vertical dotted line is the edge of the first Brillouin zone.

### 3 THE EFFECT OF CUBIC NONLINEARITY

The simplest way to describe the effect of a cubic nonlinearity is through an intensity-dependent refractive index, as illustrated in Chap. 4 [see Eq. (1-6) of Chap. 4]:  $n = n_0 + n_2|E_p|^2$ , where  $E_p$  denotes the optical field phasor. Assume now that in a localized region of the periodic structure shown in Fig. 1, the optical field is higher than that of the surrounding regions. Then for a positive  $n_2$ ,  $\omega_0$  changes to  $\omega_0' = k_0c/n < \omega_0$  in the region where the optical field is high, and hence the “nonlinear” reflectivity graph in this localized region (see Fig. 3) is a left-shifted version of the linear reflectivity graph shown in the same diagram. However, a region of high optical intensity with frequency  $\omega_0' + \Delta\omega/2$ , which can propagate in the high-intensity region, will be reflected as soon as it encounters a low-intensity region on either side, resulting in a



**Figure 3** Reflectivity as a function of frequency for a one-dimensional periodic structure. The solid line is the linear reflectivity curve, while the dashed line represents the intensity-dependent response of a nonlinear one-dimensional periodic structure, assuming  $n_2 > 0$ .

pulse that is contained in space. Thus such pulses may have soliton properties and essentially exists because of a balance between nonlinearity and the topological dispersion of the bandgap structure. Therefore these solitons are called *gap solitons* (Chen and Mills, 1987).

To model wave propagation in the presence of cubic nonlinearity, we start from the wave equation similar to Eq. (3-1) of Chap. 6:

$$\frac{\partial^2 E}{\partial z^2} - \frac{\epsilon_r(z)}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{\chi^{(3)}}{c^2} \frac{\partial^2 E^3}{\partial t^2}. \quad (3-1)$$

Now writing  $E$  in the form Eq. (2-5), the nonlinear term on the right-hand side of Eq. (3-1) can be written as

$$-\frac{3\omega_0^2 \chi^{(3)}}{8c^2} \left[ \left( |E_{e+}|^2 + 2|E_{e-}|^2 \right) E_{e+} \exp - jk_0 z + \left( |E_{e-}|^2 + 2|E_{e+}|^2 \right) E_{e-} \exp + jk_0 z \right].$$

Substituting in Eq. (3-1) and following the same steps as used to derive Eq. (2-6), we can obtain the nonlinear set of coupled equations for the forward and backward propagating envelopes as

$$\begin{aligned} \partial E_{e+} / \partial z + (n_0/c) \partial E_{e+} / \partial t + j\kappa E_{e-} + j\Gamma \left( |E_{e+}|^2 + 2|E_{e-}|^2 \right) E_{e+} &= 0 \\ -\partial E_{e-} / \partial z + (n_0/c) \partial E_{e-} / \partial t + j\kappa E_{e+} + j\Gamma \left( 2|E_{e+}|^2 + |E_{e-}|^2 \right) E_{e-} &= 0 \end{aligned} \quad (3-2)$$

where  $\Gamma \propto \chi^{(3)}$  represents the cubic nonlinearity parameter. The nonlinear terms in Eq. (3-2) represent the effects of *self* and *cross phase modulation* during the spatiotemporal evolution of the forward and backward propagating envelopes. Note that by assuming variations of  $E_{e+}, E_{e-}$  of the form as shown in Eq. (2-8), it is possible to find *amplitude-dependent dispersion relations* that are often used to model a cubically nonlinear, dispersive system. Generalized amplitude-dependent dispersion relations are used to derive nonlinear PDEs that can give rise to soliton solutions.

### 3.1 Soliton Solutions

The solution of Eqs. (3-2) is rather involved and will be briefly presented here. We follow the procedure of Aceves and Wabnitz (1989) who found the solution of the system using the solutions of a similar system called the *Thirring model*. First, to recast the equations in the form in Aceves and Wabnitz, we rescale the dependent and independent variables in Eq. (3-2) according to the normalizations

$$E_{e+} = a_+ e_+, \quad E_{e-} = a_- e_-, \quad z = bZ, \quad t = dT \quad (3-3a)$$

where

$$a_+ = a_- = (\kappa/2\Gamma)^{1/2}, b = 1/\kappa, d = n_0/c\kappa \quad (3-3b)$$

to recast the set of equations [Eq. (3-2)] in the canonical form

$$\begin{aligned} \partial e_+/\partial Z + \partial e_+/\partial T + j e_- + j(\sigma|e_+|^2 + |e_-|^2)e_+ &= 0 \\ -\partial e_-/\partial Z + \partial e_-/\partial T + j e_+ + j(|e_+|^2 + \sigma|e_-|^2)e_- &= 0 \end{aligned} \quad (3-4)$$

where  $\sigma = 1/2$  in our case. In the Thirring model, the parameter  $\sigma=0$ , implying that only cross-phase modulation terms are present in the coupled equations. To find the solution for the general case where  $\sigma \neq 0$ , assume

$$e_{\pm} = \alpha \psi_{\pm}(Z, T) \exp -j\theta(\zeta) \quad (3-5)$$

where  $\alpha$  is an unknown constant, to be determined later, and  $\theta$  is the phase, dependent on  $\zeta = (Z - VT)/\sqrt{1 - V^2}$ ,  $|V| < 1$ . Also,  $\psi_{\pm}(Z, T)$  is the one-soliton solution to the Thirring model, derived by Kaup and Newell (1977). Substitution of Eq. (3-5) in Eq. (3-4) gives two equations for  $\theta$ , which should be simultaneously satisfied:

$$\begin{aligned} d\theta/d\zeta &= (\sigma\alpha^2 \frac{1+V}{1-V} + (\alpha^2 - 1)) \sin^2 Q |\sec h(\zeta \sin Q + jQ/2)|^2 \\ &= -(\sigma\alpha^2 \frac{1-V}{1+V} + (\alpha^2 - 1)) \sin^2 Q |\sec h(\zeta \sin Q + jQ/2)|^2 \end{aligned} \quad (3-6)$$

where  $Q$  is a free parameter. This leads to the condition that

$$\alpha = \left( \frac{1 - V^2}{(1 - V^2) + \sigma(1 + V^2)} \right)^{1/2}. \quad (3-7)$$

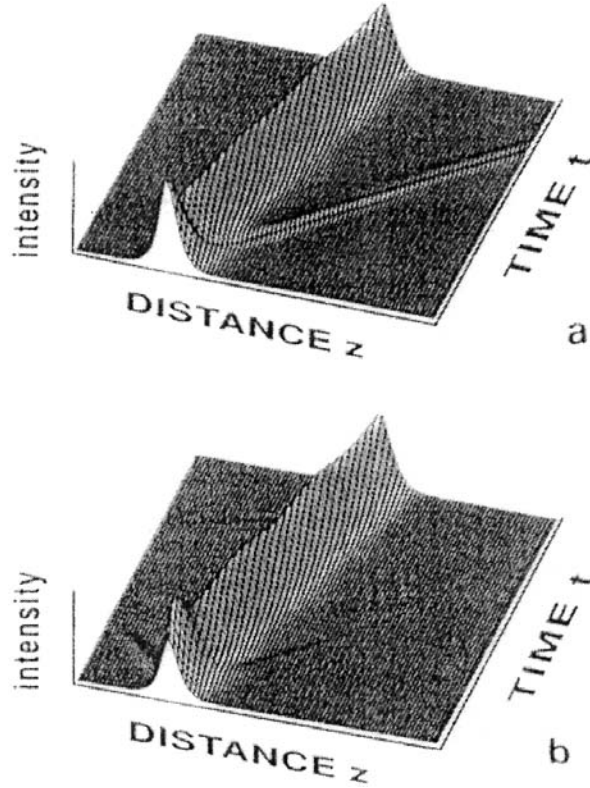
Now substituting Eq. (3-7) and one integration, we can find  $\theta$  as

$$\theta = -\frac{4\sigma V\alpha^2}{1 - V^2} \tan^{-1} [|\cot(Q/2)| \coth(\zeta \sin Q)], \quad 0 < Q < \pi, \quad |V| < 1. \quad (3-8)$$

Finally employing the known one-soliton solution of the Thirring model, the solutions of the coupled system [Eq. (3-4)] become

$$\begin{aligned} e_{\pm} &= \pm \alpha \left( \frac{1 \pm V}{1 \mp V} \right)^{1/4} \\ &\times \sin Q \exp \left( j \frac{t - Vz}{\sqrt{1 - V^2}} \cos Q - j\theta(\zeta) \right) \sec h(\zeta \sin Q + jQ/2). \end{aligned} \quad (3-9)$$

The solutions are characterized by two parameters  $Q$  and  $V$ , which determine the pulse width and velocity during propagation. Aceves and Wabnitz (1989) call these the *grating self-transparency (GST) solitons*. Typical solutions, taken from Aceves and Wabnitz (1989), are plotted in Fig. 4. The solutions



**Figure 4** Evolution of  $e_{\pm}$  for an initial condition that does not exactly match the expression in Eq. (3-9) (Aceves and Wabnitz, 1989). Note the initial radiation loss before the solutions evolve to solitons.

indicate that the solutions are stable, because for initial conditions that were not exactly in the shape predicted by Eq. (3-9), the numerical solutions evolved into the shapes predicted by Eq. (3-9) with propagation after initial radiation of the excess power, a property that is common to all soliton solutions. For small  $Q$  and  $V$ , the solutions in Eq. (3-9) can be shown to reduce to the familiar one-soliton solution of the *nonlinear Schrodinger (NLS) equation*.

### 3.2 Hysteresis and Bistability

In Chap. 5, we discussed that normal incidence of a plane wave at the interface between a linear nondispersive region and a nonlinear dispersive region medium can cause a hysteretic behavior resulting from bistability. The plots of the transmission at a certain distance in the nonlinear dispersive region as a

function of the incident intensity is plotted in Fig. 11 of Chap. 5. The dispersion was modeled to be of the waveguide type, and is included in Eq. (3-5) of Chap. 5, which is also called a modified *nonlinear Klein–Gordon equation*. Therefore it seems plausible to surmise that a similar behavior is to be expected from transmission within a nonlinear photonic bandgap structure, particularly because the photonic bandgap gives rise to a waveguide-type dispersion.

To quantitatively assess the transmission through a slice of a one-dimensional nonlinear photonic bandgap structure, consider the arrangement shown in Fig. 5, where a plane wave is incident at  $z = -L$  in a slice of the nonlinear material of length  $L$ . We need to solve Eq. (3-2) in the form

$$E_{e\pm}(z, t) = A_{\pm}(z) \exp j\delta ct/n_0 \quad (3-10)$$

subject to the boundary conditions

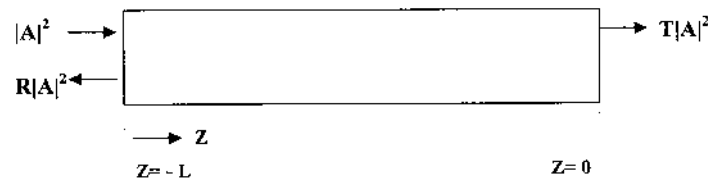
$$E_{e+}(-L, t) = A, \quad E_{e-}(0, t) = 0. \quad (3-11)$$

The *transmittivity*  $T$  is defined as

$$T = |A_+(0)/A|^2. \quad (3-12)$$

The traditional method for solving Eq. (3-2) would involve substituting Eq. (3-10) into the coupled set of equations and expressing  $A_{\pm}(z)$  into its amplitude and phase. This, in principle, would give rise to four coupled ordinary differential equations (ODEs), which need to be simultaneously solved. As we have seen in Chap. 3 on second harmonic generation, it is convenient to find conserved quantities for the system, and reexpress the solution in terms of the conserved quantities. Here we follow a simpler method used by Winful et al (1979), which involves defining four new real variables  $A_0(z)$ ,  $A_1(z)$ ,  $A_2(z)$ ,  $A_3(z)$  as

$$\begin{aligned} A_0(z) &= |A_+|^2 + |A_-|^2; & A_1(z) &= A_+ A_-^* + A_+^* A_-; \\ A_2(z) &= j(A_+^* A_- - A_+ A_-^*); & A_3(z) &= |A_+|^2 - |A_-|^2 \end{aligned} \quad (3-13)$$



**Figure 5** Schematic of setup used for calculation of stationary solutions to the coupled mode equations [Eq. (3-2)].

which also contain the information of the amplitudes and phases of the interacting waves. Note that  $A_0^2(z) = A_1^2(z) + A_2^2(z) + A_3^2(z)$ .

Upon substituting Eq. (3-13) into Eq. (3-2), and after extensive algebra, it is possible to derive the set of four coupled ODEs describing the spatial evolution of  $A_0(z)$ ,  $A_1(z)$ ,  $A_2(z)$ ,  $A_3(z)$  as:

$$\begin{aligned} dA_0/dz &= -2\kappa A_2, \\ dA_1/dz &= 2\delta A_2 + 3\Gamma A_0 A_2, \\ dA_2/dz &= -2\delta A_1 - 2\kappa A_0 - 3\Gamma A_0 A_1, \\ dA_3/dz &= 0. \end{aligned} \quad (3-14)$$

The last of the equation above suggests that

$$A_3(z) = |A_+|^2 - |A_-|^2 = \text{const} = A_3(0). \quad (3-15)$$

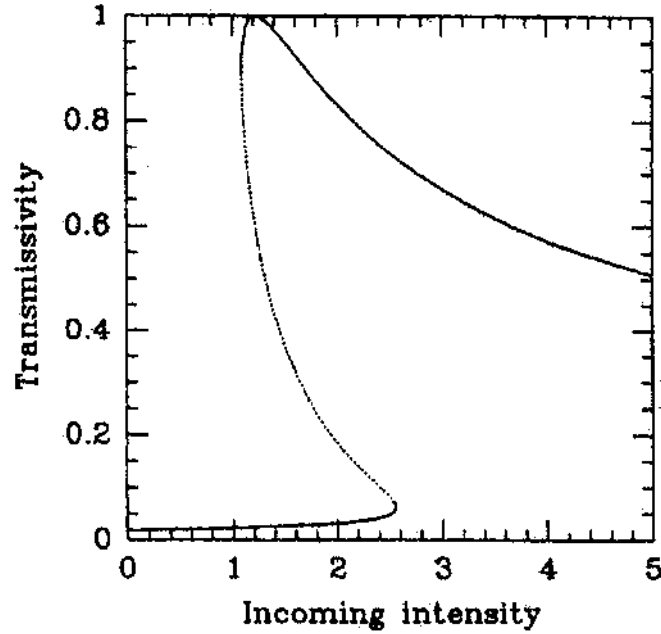
This is a conserved quantity. Note that there is another conserved quantity of the system of equations [Eq. (3-4)] above, viz.,  $\delta A_0 + \kappa A_1(z) + 3\Gamma A_0^2/4$ . Using these conserved quantities, we can simplify Eq. (3-14) to yield one ODE for  $A_0(z)$ :

$$\begin{aligned} dA_0(z)/dz &= \pm \sqrt{(A_0 - A_3(0))[\kappa^2(A_0 + A_3(0)) - (A_0 - A_3(0))(\delta + 3\Gamma((A_0 + A_3(0)))^2)]} \\ &= \pm \sqrt{D(A_0)}. \end{aligned} \quad (3-16)$$

From the boundary conditions, we require  $A_0(0) = A_3(0)$ . The standard procedure for solving equations such as Eq. (3-16) is to recast it in the form (Martijn de Sterke and Sipe, 1994)

$$\pm \int \frac{dA_0(z)}{\sqrt{D(A_0)}} = z + \text{const}. \quad (3-17)$$

The solutions are expressible in terms of *Jacobian elliptic functions*, similar to what was done in Sec. 2.4 of Chap. 3 for SHG for depleted pump and phase mismatch. In the case of Eq. (3-17), there are four roots of the denominator, of which one is  $A_0 = A_3(0)$ . The other roots must be found by solving the cubic polynomial in square brackets in Eq. (3-16). The exact solution depends on the values of the four roots of the polynomial  $D(A_0)$ , and will not be explicitly written here for the sake of simplicity. Suffice to state that the solutions are in general periodic in nature, because the elliptic functions such as  $sn^2$  are periodic *w.r.t. z*. If the period exactly equals the length of the system, then the optical fields at the front and back of the system are identical. This implies that the material appears transparent to the incident field, and the transmittivity is unity.



**Figure 6** Typical example showing hysteresis behavior for the transmittivity as a function of the incident intensity (Martijn de Sterke and Sipe, 1994).

From Eq. (3-17), one can compute the value of  $A_0(0)$  and hence the value of  $T$  with  $A_3(0)$  as a parameter. However,  $A_3(0)$  is related to  $T$  through the relation  $A_3(0) = T|A|^2$ . This makes the final relation between  $T$  and  $|A|^2$  multivalued. A typical sketch of the variation of  $T$  vs.  $|A|^2$  is shown in Fig. 6. As stated in Chap. 5, multivaluedness is a necessary condition for *hysteresis* and *bistability*. It can be rigorously shown that the part of the graph with a negative gradient (dotted line) is unstable, while those parts with positive gradient are stable.

#### 4 SECOND HARMONIC GENERATION IN PHOTONIC BANDGAP STRUCTURES

In Chap. 3, we have introduced the basic mechanism for second harmonic generation (SHG), and have studied SHG for the cases of undepleted and depleted pumps, as well as for perfect phase matching and the phase mis-

matched case. One of the ways of ensuring phase matching as mentioned in Chap. 3 is to employ the anisotropy of the crystal. Another technique used by researchers is to alternate layers or stacks of oppositely poled nonlinear material. The thickness of each layer is proportional to  $\Gamma\zeta/2$  in Eq. (2-39) of Chap. 3. The *periodic poling* periodically changes the sign of the second-order nonlinear coefficient  $\chi^2$  (Armstrong et al., 1962; McMullen, 1975). After the first layer, assuming zero initial second harmonic, there is some generation of the second harmonic and consequent depletion of the fundamental. The relative phase difference between the fundamental and the second harmonic is approximately  $\pi/2$ , and the generated second harmonic is at the peak of what is attainable in the regular phase mismatched case. This forms the initial conditions for the second layer where the sign of the nonlinearity is opposite to that of the first layer. It can be shown from Eq. (2-30) of Chap. 3 along with the redefined normalizations given by Eq. (2-17) of Chap. 3 (because  $\chi^{(2)}$  now has opposite sign) that the second harmonic further grows with propagation. This process continues at each layer until finally one attains overall amplification of the second harmonic much beyond the maximum predicted value for the case of a bulk phase-mismatched medium.

As an alternative to engineering the nonlinear properties of the constituent layers, one can also attempt to alter the *topological dispersion* of the medium to cancel the effects of the linear phase mismatch. The idea is to introduce a periodic modulation of the linear refractive index of the medium, to induce a phase-matching condition for SHG, as can be introduced using a uniform *Bragg grating*. Along the same lines, this can be achieved through the use of photonic bandgap structures such as the type discussed in this chapter. The concept was suggested by Bloembergen and Sievers (1970), and the first experimental demonstration of reflected second harmonic enhancement was accomplished with counterpropagating beams in a GaAs–GaAlAs structure (van der Ziel and Ilegems, 1976). The fundamental beam was tuned to the center of the first bandgap, as shown in Fig. 2b, where maximum reflection is expected to occur. The second harmonic was centered around the middle of the passband between two stop-bands, allowing maximum transmission of the second harmonic.

To quantify the generation of second harmonic in a one-dimensional photonic bandgap structure, we assume a truncated Fourier series expansion of the relative permittivity, as described in Eqs. (2-2) and (2-3) in the form

$$\varepsilon_r(z; \omega) = n_0^2(\omega) + 2\varepsilon_1 \cos K_g z + 2\varepsilon_2 \cos K_g z. \quad (4-1a)$$

Equation (4-1a) is a more general form of Eq. (2-4) where one more term in the Fourier series expansion of  $\Delta\varepsilon$  has been included. Furthermore, because we

are analyzing the propagation of two frequencies, viz., the fundamental and the second harmonic, we define

$$n_0^2(\omega_0) = n_{10}^2, \quad n_0^2(2\omega_0) = n_{20}^2 \quad (4-1b)$$

and assume

$$\chi^{(2)}(z) = \chi_0^{(2)} + 2\chi_1^{(2)} \cos K_g z \quad (4-1c)$$

as in De Angelis et al. (2001).

In what follows, we will derive the evolution of the second harmonic under the assumption of an undepleted pump (see Chap. 3). This will be achieved by first solving for the spatial distribution of the fundamental, followed by the derivation of the evolution of the second harmonic.

Recall that in Sec. 2, we assumed the optical field to have a propagation constant equal to  $k_0 = \pi/d$ , because we were interested in obtaining the dispersion relation around the frequency where maximum reflection was expected. In the present case, we will be interested in arbitrary frequencies of the fundamental  $\omega_0$  (and hence the second harmonic  $2\omega_0$ ) to find the optimum frequency that may give rise to maximum conversion efficiency. Accordingly, we will now assume forward and backward traveling waves for the fundamental in the form

$$E_1(z, t) = (1/2)[E_{e1+}(z)\exp j(\omega_0 t - k_1 z) + E_{e1-}(z)\exp j(\omega_0 t + k_1 z) + c.c.] \quad (4-2)$$

where the subscript 1 designates the fundamental at frequency  $\omega_0$ . Substitution of Eqs. (4-1a) and (4-1b) with Eq. (4-2) into the wave equation [Eq. (2-1)], and upon separating the coefficients of  $\exp j(\omega_0 t \pm k_1 z)$ , leads to a pair of coupled mode equations for the spatial evolution of the forward and backward traveling fields. Note that we consider the linear wave equation rather than the nonlinear wave equation (see Chap. 2) for the propagation of the fundamental because of the *undepleted pump approximation*. The coupled equations are:

$$\begin{aligned} dE_{e1+}/dz &= -j\kappa_1 E_{e1-} \exp(j\Delta_1 z) \\ dE_{e1-}/dz &= j\kappa_1 E_{e1+} \exp(-j\Delta_1 z) \end{aligned} \quad (4-3)$$

where  $\Delta_1 = 2k_1 - K_g$  and  $\kappa_1 = \omega_0 \epsilon_1 / 2n_{10} c$ . Equation (4-3) can be conveniently solved by first rewriting  $E_{e1+} = \tilde{E}_{e1+} \exp(j\frac{\Delta_1}{2} z)$ ;  $E_{e1-} = \tilde{E}_{e1-} \exp(-j\frac{\Delta_1}{2} z)$ , and substituting into Eq. (4-3) to obtain

$$\begin{aligned} d\tilde{E}_{e1+}/dz + j\frac{\Delta_1}{2} \tilde{E}_{e1+} &= -j\kappa_1 \tilde{E}_{e1-}; \\ d\tilde{E}_{e1-}/dz - j\frac{\Delta_1}{2} \tilde{E}_{e1-} &= j\kappa_1 \tilde{E}_{e1+}. \end{aligned} \quad (4-4)$$

Now we can use Laplace transforms to solve the set of simultaneous ODEs with constant coefficients. The solutions are of the form

$$E_{e1+}(z) = \left[ \tilde{E}_{e1+}(0) \cos \Phi_1 z - j \frac{(\Delta_1/2) \tilde{E}_{e1+}(0) + \kappa_1 \tilde{E}_{e1-}(0)}{\Phi_1} \sin \Phi_1 z \right] \exp j(\Delta_1/2)z;$$

$$E_{e1-}(z) = \left[ \tilde{E}_{e1-}(0) \cos \Phi_1 z + j \frac{(\Delta_1/2) \tilde{E}_{e1-}(0) + \kappa_1 \tilde{E}_{e1+}(0)}{\Phi_1} \sin \Phi_1 z \right] \exp -j(\Delta_1/2)z;$$

$$\Phi_1^2 = (\Delta_1/2)^2 - \kappa_1^2. \quad (4-5)$$

Finally, we have to invoke the boundary conditions.  $E_{e1+}(z=0) = E_0$ ,  $E_{e1-}(z=L)=0$ , where  $L$  denotes the length of the grating, to solve for  $\tilde{E}_{e1+}(0), \tilde{E}_{e1-}(0)$ . After some algebra, it follows that

$$\tilde{E}_{e1+}(0) = E_0, \quad E_{e1-}(0) = -j \frac{(\kappa_1/\Phi_1) \sin \Phi_1 L}{\cos \Phi_1 L + j \frac{\Delta_1/2}{\Phi_1} \sin \cos \Phi_1 L} E_0. \quad (4-6)$$

To obtain the largest second harmonic field, it is intuitive to have the maximum pump or fundamental field in the photonic bandgap structure. As shown by Centini et al (1999), this happens when  $\Phi_1 L = m\pi$ , where there is a *transmission resonance*. This is similar to the case of a Fabry–Perot cavity where there is maximum stored energy at a transmission maximum. Using  $m = 1$ , the solutions for the forward and backward traveling fundamental waves become

$$E_{e1+}(z) = E_0 \left[ \cos \Phi_1 z - j \frac{\Delta_1/2}{\Phi_1} \sin \Phi_1 z \right] \exp(j\Delta_1/2z)$$

$$= \tilde{E}_{e1+}(z) \exp(j\Delta_1/2z), \quad (4-7)$$

$$E_{e1-}(z) = jE_0\kappa_1 \sin \Phi_1 z \exp(-j\Delta_1/2z) \equiv \tilde{E}_{e1-}(z) \exp(-j\Delta_1/2z).$$

Note that from Eq. (4-7),  $|E_{e1+}(z)|^2 - |E_{e1-}(z)|^2 = E_0^2$ , a constant, as is usual for interacting waves in a reflection grating. It is interesting to note that for the forward traveling fundamental wave, the transmission is 100% at  $z = L$ . However, for the backward propagating fundamental, the intensities at  $z = 0, L$  are equal to zero, with a maximum at  $z = L/2$ . The *distributed feedback* provided by the grating allows the power of the fundamental to be efficiently stored inside the photonic bandgap structure.

If now one uses the nonlinear wave equation for the second harmonic as in Eq. (2-1) of Chap. 3, with the nonlinear polarization acting as the driving term in the equation coming from the square of the fundamental field, one can

derive a set of coupled ODEs for the forward and backward propagating components of the second harmonic. Accordingly, we define

$$E_2(z, t) = (1/2)[E_{e2+}(z)\exp j(2\omega_0 t - k_2 z) + E_{e2-}(z)\exp j(2\omega_0 t + k_2 z) + c.c.] \quad (4-8)$$

where, by analogy with Eq. (4-7), we a priori set

$$E_{e2+}(z) = \tilde{E}_{e2+}(z)\exp j\frac{A_2}{2}z; E_{e2-}(z) = \tilde{E}_{e2-}(z)\exp -j\frac{A_2}{2}z. \quad (4-9)$$

In Eq. (4-9),  $A_2 = 2k_2 - 2K_g$ . Upon substituting Eq. (4-7) and (4-8) into the wave equation for the second harmonic, and using Eq. (4-1c) for the nonlinear susceptibility, one obtains the following set of coupled ODEs for the forward and backward propagating second harmonic fields (De Angelis et al., 2001):

$$\begin{aligned} d\tilde{E}_{e2+}/dz + j\frac{A_2}{2}\tilde{E}_{e2+} + j\kappa_2\tilde{E}_{e2-} &= -j(\omega_0/cn_{20})\left[\chi_0^{(2)}\tilde{E}_{e1+}^2 + 2\chi_1^{(2)}\tilde{E}_{e1+}\tilde{E}_{e1-}\right] \\ d\tilde{E}_{e2-}/dz - j\frac{A_2}{2}\tilde{E}_{e2-} - j\kappa_2\tilde{E}_{e2+} &= j(\omega_0/cn_{20})\left[\chi_0^{(2)}\tilde{E}_{e1-}^2 + 2\chi_1^{(2)}\tilde{E}_{e1+}\tilde{E}_{e1-}\right] \end{aligned} \quad (4-10)$$

where  $\kappa_2 = \omega_0\epsilon_2/n_{20}c$ , and where the phase matching condition, viz.,  $\Phi_2 = 2\Phi_1$  is assumed to be satisfied. With the left hand side (LHS) of the above equations set equal to zero (i.e., no driving term), the equations are similar to the corresponding ODEs for the evolution of the fundamental as in Eq. (4-4). The requisite boundary conditions for solving the set of ODEs above are  $E_{e2+}(z = 0) = 0$ ,  $E_{e2-}(z = L) = 0$ , implying zero second harmonic amplitudes for the forward and backward traveling components at  $z = 0$  and  $z = L$ , respectively.

The solutions to Eq. (4-10) using Eq. (4-7) to substitute for the driving terms is complicated, and can be found in De Angelis et al (2001). We quote here a result from their work, which relates the forward traveling second harmonic amplitude to the amplitude of the incident fundamental field. It can be shown that for transmission resonance of the second harmonic field, the second harmonic amplitude can be expressed as

$$\begin{aligned} E_{e2+}(z = L) &= \frac{E_0^2 L \omega_0}{4cn_{20}} \left[ \chi_0^{(2)} \left( 1 + \frac{A_1^2 + A_1 A_2}{4\pi^2} L^2 \right) + \chi_1^{(2)} \frac{\kappa_1 L}{2\pi^2} (2\kappa_2 - A_2 - 2A_1) \right] \\ &= E_0^2 L \chi_{\text{eff}}^{(2)} \end{aligned} \quad (4-11)$$

where  $\chi_{\text{eff}}^{(2)}$  depends on  $\chi_0^{(2)}$ ,  $\chi_1^{(2)}$  and the parameters  $\kappa_1$ ,  $\kappa_2$ ,  $A_1$ ,  $A_2$  defined above. In other words, strong enhancement of the second harmonic can be achieved by the confinement of the fundamental within the structure, which in turn is related to the depth of modulation of the refractive index. Typical

results are plotted in De Angelis et al (2001) for various combinations of  $\kappa_1$ ,  $\kappa_2$ ,  $\Delta_1$ ,  $\Delta_2$ .

A lot of recent work has been done in connection with enhanced SHG using photonic bandgap structures. Interested readers are referred to, for instance, the work of Scalora et al (1997) for pulsed SHG and Saltiel and Kivshar (2000) for phase matching in *higher-dimensional photonic bandgap structures*.

## 5 PROBLEMS

1. Show that the exact dispersion relation for wave propagation in a one-dimensional periodic layered structure comprising regions 1 and 2 with permittivities  $\epsilon_{1,2}$ , and of thicknesses  $d_{1,2}$  with  $d_1 + d_2 = d$  (see Fig. 1) is given by

$$\cos kd = \frac{(k_1/k_2 + 1)^2}{4k_1/k_2} \cos(k_1 d_1 + k_2 d_2) - \frac{(k_1/k_2 - 1)^2}{4k_1/k_2} \cos(k_1 d_1 - k_2 d_2)$$

where  $k_{1,2}$  are the propagation constants of light in regions 1 and 2, respectively, related to the frequency of the wave, and  $k$  denotes the wave number of the propagating wave through the periodic layered structure.

2. (a) Starting from the nonlinear Schrodinger equation as in Eq. (3-2) of Chap. 8 with  $\alpha = 0$  determine the amplitude-dependent dispersion relation. To do this, assume the dependent variable to be of the form  $a \exp(\Omega t - Kz)$ , where  $\Omega, K$  denote excursions of the frequency and wave number around the carrier  $\omega_0, k_0$ . Hence determine the nonlinearity coefficient defined as  $\partial \omega / \partial a^2$ .  
(b) Starting from the coupled set of equations as in Eq. (3-2), determine the amplitude-dependent dispersion relation for this case.
3. Using Eq. (4-3) as a starting point, determine the transmittivity and reflectivity for an incident CW optical field of frequency  $\omega_0$  onto a one-dimensional periodic structure. The transmittivity and reflectivity are defined as  $T_{\text{eff}} = |E_{e1+}(L)/E_{e1+}(0)|^2$ ,  $R_{\text{eff}} = |E_{e1-}(0)/E_{e1+}(0)|^2$ . Plot the transmittivity and reflectivity as a function of the phase mismatch parameter. Hence plot the reflectivity as a function of the frequency  $\omega_0$  of the incident wave. Compare your plot with Fig. 3.
4. Describe the dispersion relation in Fig. 2a by means of a Taylor series expansion around an operating frequency  $\omega_0$ . Hence find the

underlying linear partial differential equation for a forward traveling wave. Add the effect of a cubic nonlinearity, and show that the resulting partial differential equation is the nonlinear Schrodinger equation. Write down an one-soliton solution for this equation.

5. Analyze second harmonic generation in a one-dimensional photonic bandgap structure without the assumption of a nondepleted fundamental (pump). To do this, first set up the evolution equations of the forward and backward traveling pumps and second harmonics, and thereafter numerically solve the coupled set of equations.

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# Appendix A

## The Split Step Beam Propagation Method

If we wish to consider propagation in a material where the propagation constant or equivalently the refractive index is a function of position, either due to profiling of the material itself (such as a *graded index fiber* or a *grating*) or due to induced effects such as third-order nonlinearities, the paraxial wave equation changes to

$$\partial E_e / \partial z = \frac{1}{2jk_0} \nabla_{\perp}^2 E_e - j\Delta n k_0 E_e. \quad (1-1)$$

The quantity  $\Delta n$  is the change in the refractive index over the ambient refractive index  $n_0 = c/v$ , where  $c$  is the velocity of light in vacuum. Eq. (1-1) is a modification of Eq. (4-5) in Chapter 1 and can be derived from the scalar wave equation when the propagation constant, or equivalently the velocity of the wave, is a function of  $(x, y, z)$  explicitly as in gratings or fibers, or implicitly such as through the intensity-dependent refractive index.

The paraxial propagation equation (Eq. (1-1)) is a partial differential equation (PDE) that does not always lend itself to analytical solutions, except for some very special cases involving special spatial variations of  $\Delta n$ , or when, as in nonlinear optics, one looks for particular soliton solution of the resulting nonlinear PDE using exact integration or inverse scattering methods. Numerical approaches are often sought for to analyze beam (and pulse) propagation in complex systems such as optical fibers, volume diffraction gratings, Kerr and photorefractive media, etc. A large number of numerical methods can be used for this purpose. The *pseudospectral methods* are often favored over *finite difference methods* due to their speed

advantage. The *split-step beam propagation method* (BPM) is an example of a pseudospectral method.

To understand the philosophy behind the BPM, it is useful to rewrite Eq. (1-1) in the form (Agrawal, 1989; Jarem and Banerjee, 2001; Poon and Banerjee, 2002)

$$\partial E_e / \partial z = (\hat{D} + \hat{S}) E_e \quad (1-2)$$

where  $\hat{D}, \hat{S}$  are a linear differential operator and a space-dependent or nonlinear operator, respectively [see, for instance, the structure of Eq. (1-2)]. Thus, in general, the solution of Eq. (1-2) can be symbolically written as

$$E_e(x, y, z + \Delta z) = \exp[(\hat{D} + \hat{S})\Delta z] E_e(x, y, z) \quad (1-3)$$

if  $\hat{D}, \hat{S}$  are assumed to be  $z$ -independent. Now for two noncommuting operators  $\hat{D}, \hat{S}$ ,

$$\exp(\hat{D}\Delta z)\exp(\hat{S}\Delta z) = \exp(\hat{D}\Delta z + \hat{S}\Delta z + (1/2)[\hat{D}, \hat{S}](\Delta z)^2 + \dots) \quad (1-4)$$

according to the *Baker–Hausdorff formula*, where  $[\hat{D}, \hat{S}]$  represents the commutation of  $\hat{D}, \hat{S}$ . Thus up to second order in  $\Delta z$ ,

$$\exp(\hat{D}\Delta z + \hat{S}\Delta z) \cong \exp(\hat{D}\Delta z)\exp(\hat{S}\Delta z) \quad (1-5)$$

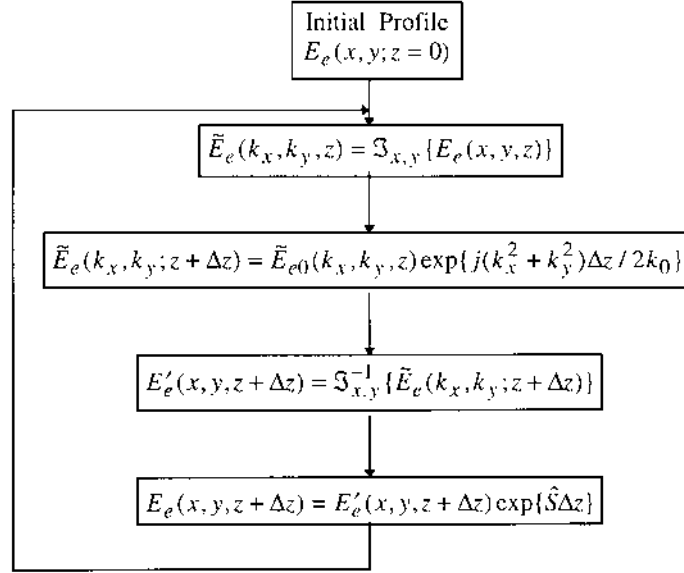
which implies that in Eq. (1-1) the diffraction and the inhomogeneous operators can be treated independent of each other.

The action of the first operator on the RHS of Eq. (1-5) is better understood in the spectral domain. Note that this is the propagation operator that takes into account the effect of diffraction between planes  $z$  and  $z + \Delta z$ . Propagation is readily handled in the spectral or spatial frequency domain using the transfer function for propagation written in Eq. (4-9) of Chapter 1 with  $z$  replaced by  $\Delta z$ . The second operator describes the effect of propagation in the absence of diffraction and in the presence of medium inhomogeneities, either intrinsic or induced, and is incorporated in the spatial domain. A schematic block diagram of the BPM method in its simplest form is shown in Fig. 1. There are other modifications to the simple scheme, viz., the symmetrized split-step Fourier method, and the leap-frog techniques, these are discussed in detail elsewhere (Agrawal, 1989).

## 1 EXAMPLES

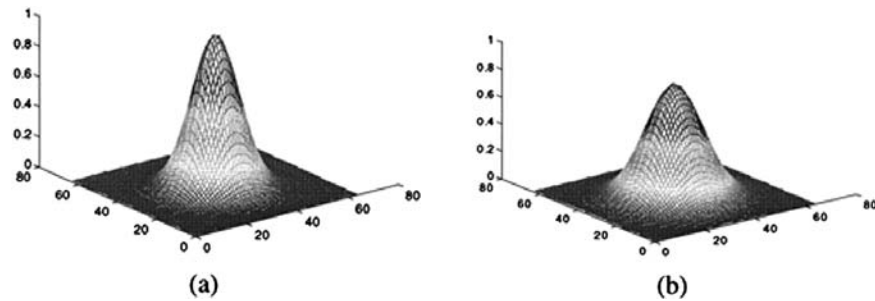
### 1. Linear Free-Space Propagation

In this case, the inhomogeneous operator is zero, and we can solve Fresnel diffraction of beams using the BPM method. Of course, propagation from a plane  $z=0$  to arbitrary  $z$  can be performed in one step in this



**Figure 1** Flow diagram for the BPM split step method.

case; however, in the example we provide, we use the split-step method to convince readers that the result is identical to what one would get if the propagation was covered in one step. In Fig. 2, we show the profile of a diffracted Gaussian beam after propagation through free-space, and the results agree with the physical intuition of increased width and decreased on-axis amplitude during propagation.



**Figure 2** Diffraction of a Gaussian beam during free space propagation. (a) Profile at  $z = 0$  (plane wavefronts assumed), (b) profile at  $z = z_R$ , where  $z_R$  is the Rayleigh length of the original Gaussian beam.

**Table 1** MATLAB Code Used to Study Propagation of Gaussian Beam Through a Kerr Medium

---

```

clear
% The following two lines define the 2-d grid in space.
x1 = [-4.:8./64.:-4 + 63.*8/64.];
x2 = [-4.:8./64.:-4 + 63.*8/64.];
% delz is the step size, and N is the number of steps.
delz = 1.0;
N = 35;
% y is the initial 2-d Gaussian beam.
y = 2.01*(exp(-2.*x1.*x1))*(exp(-2.*x2.*x2)) + 0.000*rand(64);
% The energy statements below and in the last line is to check for
% numerical accuracy.
energy = sum(sum(abs(y.^2)))
figure (1), mesh (x1,x2,abs(y)), view (90,0)
% The following two lines define the 2-d grid in spatial frequency.
u1 = [-1.:2./64.:-1. + 63.*2./64.];
u2 = [-1.:2./64.:-1. + 63.*2./64.];
% v is the transfer function for propagation.
v = (exp(i*u1.*u1*delz*0.05))*(exp(i*u2.*u2*delz*0.05));
% Shifting is required to properly align the FFT of y, and w for multiplication.
w = fftshift(v);
% The part from for to end in the "DO" loop is a split-step
% beam propagation method where medium nonlinearities can be
% incorporated in the spatial domain in the variable p below.
for j = 1:N;
z = fft2(y);
zp = z.*w;
yp = ifft2(zp);
yint = (yp.*conj(yp) + conj(yp).*yp)/2.0;
p = exp(-i*yint*0.05);
%p = 1;
yp = yp.*p;
zp = fft2(yp);
zp = zp.*w;
yp = ifft2(zp);
y = yp;
end
figure (2), mesh (x1,x2,abs(y)), view (90,0)
energy_p = sum(sum(abs(yp.^2)))

```

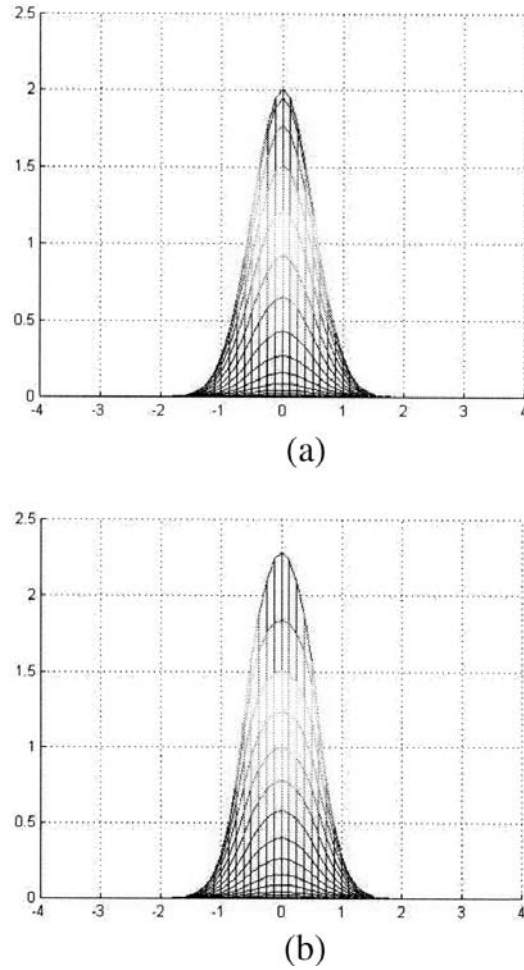
---

MATLAB OUTPUT:

```

energy = 203.0776
energy_p = 203.0776

```



**Figure 3** (a) Initial Gaussian beam shape and (b) Beam shape after propagation in the Kerr medium. Notice that self-focusing increases the peak value. Energy is conserved in the process as shown by the MATLAB output in Table 1.

## 2. Self-Focusing Of Optical Beam

As stated in Chapter 4, in general, it is not possible to find analytical solutions for the variation of a beam shape with propagation through a cubically nonlinear medium, and one has to resort to numerical techniques to analyze the propagation. An example of such a program written in MATLAB is shown in Table 1. We use the split-step beam propagation method outlined above. The plots in Fig. 3(a–b) are cross-sectional cuts of the beam

shapes in two transverse dimensions. The program can be readily modified to analyze propagation of beams with a one-dimensional transverse profile. When the beam profile becomes too steep, i.e., the width approaches the wavelength, one has to resort to specialized methods to simulate the propagation. One such technique involves using an adaptive full-wavelet transform technique, which is outlined in Appendix B.

The split-step method has been used in the book in connection with beam propagation through a Kerr-type material, photorefractive materials, liquid crystals, etc.

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# Appendix B

## Wavelet Transforms and Application to Solution of Partial Differential Equations

### 1 INTRODUCTION TO WAVELETS

Before turning to a formal definition of the *wavelet*, it will be constructive to understand the general utility that has made wavelets so popular. In contrast to *Fourier analysis*, which uses as its *basis functions* sinusoids of varying frequencies but of infinite duration, wavelet analysis uses basis functions which have limited duration (allows for *compact support*). These are limited duration waves which are referred to as ‘wavelets.’ When a set of these wavelets are used as basis functions to perform a transform from one domain (e.g., space or time) to another (i.e., wavelet), the operation becomes a wavelet transform. The process begins with the basic wavelet function or *mother wavelet*  $\psi(x)$ , followed by both a *shifting* operation  $\{\psi(x \pm k)\}$  and a *scaling* operation  $\{\psi(\frac{x}{a})\}$ , where  $k$  and  $a$  are typically nonnegative integers ( $a > 0$ ). It is not difficult to see that by shifting and scaling by the proper amount, wavelet functions can be generated which can accurately “track” very short lived signal transients, for example, sudden amplitude changes which might occur in an optical shock wave formation.

This flexibility gives wavelets time-widths which adapt to their frequency; high-frequency wavelets  $\{\psi(\frac{x+k}{a})\}$  are very narrow (small  $a$ ), while low-frequency wavelets  $\{\psi(\frac{x+k}{a})\}$  are very wide (large  $a$ ). This concept of *frequency-adapted wavelets* leads nicely to an adaptive mesh when incorporated into the proper numerical scheme (described in the next chapter). This adaptive mesh provides the flexibility to choose the level at which to compress the wavelets; this is done so only in areas of high signal transients. The reason for using this adaptive mesh is that the more a wavelet is compressed, the number of wavelets required increases proportionately, thus increasing

storage requirements and processing time. These ideas and more will be described in the following subsections.

## 2 WAVELET PROPERTIES AND SCALING FUNCTIONS

It is a fair question to begin this section by asking the question “What is a wavelet, anyway?” A particular answer to this question has already been given, but now a more formal definition of what it takes for a function to constitute a wavelet will be given. The focus here is on those functions which are *square-integrable* on the real line, or *finite-energy functions*, and denoted by the notation  $f(x) \in L^2(\mathfrak{R}) \in \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ . Thus the requirements for a function  $\psi(x) \in L^2(\mathfrak{R})$  to be a wavelet are as follows:

- i)  $\psi(x)$  is some oscillatory function which rapidly goes to zero as  $|x| \rightarrow \infty$ .
- ii)  $\psi(x)$  is a real-valued function with a Fourier spectrum  $\Psi(\omega)$  satisfying the *admissibility criteria*:

$$\int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2 d\omega}{|\omega|^2} < \infty. \quad (2-1)$$

- iii) The presence of the  $\omega^{-1}$  term in Eq. (2-1) requires that  $\Psi(0) = 0$ , or

$$\int_{-\infty}^{\infty} \psi(x) dx = 0 \Rightarrow \text{wavelet has average value of zero.} \quad (2-2)$$

- iv) Because  $\Psi(0) = 0$  and  $\Psi(\infty) = 0$ , a wavelet will have the amplitude spectrum of a bandpass filter.

If the wavelet function becomes strictly zero outside some finite interval, then requirement i) is the condition for *compact support*. While not all wavelets have compact support, it is a requirement if the wavelet basis is to form an orthogonal basis for the *wavelet transform*. Compact support for a wavelet function  $\psi(x)$  may be stated mathematically as follows:

$$\psi(x) = \begin{cases} 0, & x < x_{\min} \\ \psi(x), & x_{\min} \leq x \leq x_{\max} \\ 0, & x > x_{\max} \end{cases} \quad (\text{compact support}). \quad (2-3)$$

In addition to the four requirements listed above, there are three useful properties which give added utility to any given wavelet. These properties are (Daubechies, 1992)

- i) *symmetry*—useful for maintaining phase and symmetry properties of the original signal.
- ii) *regularity*—the order of differentiability of the wavelet, useful for providing “smoothness” in the reconstructed signal.
- iii) number of vanishing *moments*—a wavelet has  $p$  moments = 0 if

$$\int_{-\infty}^{\infty} x^j \psi(x) dx = 0 \quad \text{for } j = 0, 1, \dots, p-1. \quad (2-4)$$

This property ensures  $p$ th-order decay of the wavelet transform coefficients in areas of smoothness in  $f(x)$ . Thus wavelet coefficients are negligible except in areas of sudden change (transients) in  $f(x)$ .

Any wavelet function that satisfies the four criteria listed above can be used to form a set of *wavelet basis functions*. These basis functions, denoted  $\{\psi_{a,b}(x)\}$ , are generated by *scaling* (compressing or stretching) and *shifting* the basic wavelet function as follows:

$$\psi_{a,b}(x) = a^{\frac{1}{2}} \psi\left(\frac{x-b}{a}\right). \quad (2-5)$$

In this equation,  $a > 0$  represents the scale (or width) of a given wavelet while  $b$  describes its shifted position along the  $x$ -axis. Most often, wavelet basis functions are generated on a *dyadic scale* (i.e., scaling and shifting by powers of 2). Thus as is the case for this research, wavelet basis functions will be generated according to the *dyadic expression* (Bogess and Narcowich, 2001)

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi\left(\frac{x-k}{2^j}\right), \quad (2-6)$$

where  $j$  and  $k$  are both integers with  $j \geq 0$ ,  $-\infty < k < \infty$ .

Using dyadic wavelets results in the generation of the *discrete wavelet transform* (DWT). In the DWT, the transform coefficients are given by

$$c_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx, \quad (2-7)$$

whereas a function representation  $f(x) \in L^2(\mathfrak{R})$  is given by the wavelet series expansion

$$f(x) = \sum_j \sum_k c_{j,k} \psi_{j,k}(x). \quad (2-8)$$

While this definition of the DWT looks convenient, it is not very practical in actual use because its use requires the scale factor for  $j$  to go to infinity in order to accurately represent a function  $f(x)$ . Thus a wavelet function by itself is only useful for detecting details, or transients, in an otherwise smooth function. In order to accurately represent a function from its transform coefficients, another function, called a *scaling function*  $\phi(x)$ , should be introduced. The

relation between  $\psi(x)$ ,  $\phi(x)$ ,  $f(x)$ , and the corresponding transform coefficients are better interpreted in the context of multi-resolution analysis.

### 3 DIGITAL FILTERS AND MULTI-RESOLUTION ANALYSIS

An effective computer algorithm useful for wavelet analysis begins with the low- and high-pass digital filters which are used to create the corresponding scaling and wavelet functions through *dilation* and *recursion*. This subject leads naturally then to the concept of *multi-resolution analysis* (MRA). In essence, the scaling function comes from the *low-pass filter* (LPF) while the wavelet function comes from the *high-pass filter* (HPF). Thus the properties of the scaling and wavelet functions described earlier are reflected in the digital filters used to generate them. The main reason for this approach to wavelet analysis is that analytic expressions for the functions themselves rarely exist, but rather these continuous functions are usually obtained from the LPF and HPF through dilation and recursion (Bogess and Narcowich, 2001).

The process to generate  $\phi(x)$  begins with a *finite impulse response* (FIR) LPF designated  $\mathbf{h}(\mathbf{k})$ , which has  $L$  nonzero coefficients, combined with the dilation equation given below:

$$\phi(x) = 2 \sum_{k=0}^{L-1} \mathbf{h}(\mathbf{k}) \phi(2x - k). \quad (3-1)$$

Note that the important property of a limited duration function  $\phi(x)$  (i.e., compact support) corresponds to an FIR LPF (finite number  $L$  of filter coefficients). Thus

$\phi(x)$  has compact support  $\Leftrightarrow \mathbf{h}(\mathbf{k})$  is FIR LPF.

Also note that while the wavelet function has an average value of zero, the scaling function has an average value (normalized) of unity so that

$$\int_{-\infty}^{\infty} \phi(x) dx = 1. \quad (3-2)$$

This will be achieved if the designated filter  $\mathbf{h}(\mathbf{k})$  satisfies

$$\sum_k \mathbf{h}(\mathbf{k}) = 1. \quad (3-3)$$

Likewise, the wavelet function is generated from an FIR HPF  $\mathbf{g}(\mathbf{k})$  according to

$$\psi(x) = 2 \sum_{k=0}^{L-1} \mathbf{g}(\mathbf{k}) \phi(2x - k), \quad (3-4)$$

where the HPF is obtained from the LPF simply by reversing the order of the LPF coefficients and changing the sign on the even elements. This construction for  $\mathbf{h}(\mathbf{k})$  and  $\mathbf{g}(\mathbf{k})$  is known as a *conjugate quadrature filter* (CQF) pair and is described as

$$\mathbf{g}(\mathbf{k}) = (-1)^k \mathbf{h}(1 - k). \quad (3-5)$$

As noted earlier,  $\psi(x)$  has an average value of zero, which means

$$\sum_k \mathbf{g}(\mathbf{k}) = 0. \quad (3-6)$$

If compact support is a desired property, it is evident that both the LPF and HPF must be FIR filters. This will then ensure that both the wavelet function and the scaling function have compact support. Examples of wavelet functions that do not have compact support include the *Morlet*, *Mexican Hat*, and *Meyer* wavelets. The Morlet and Mexican Hat wavelets have explicit expressions which include an  $e^{-x^2}$  term, and thus never reach zero as  $|x| \rightarrow \infty$ . The Meyer wavelet has an explicit expression, but only in the frequency domain. Its transform in the space domain does not have compact support. Examples of wavelets that do have compact support include the *Haar*, *Daubechies*, *Coiflet*, and *Symlet* wavelets (Chui, 1992). Daubechies is credited with creating the latter three wavelet families. The *Symlet* is the wavelet which has been used to simulate propagation of solitons—its properties are described in the next section.

Finally, the concept of MRA will now be explained, the process by which the wavelet analysis will actually be performed in this research. The goal of MRA is to decompose the function space into wavelet subspaces—so that there is a piece (projection) of  $f(x)$  in each subspace. Basically, there are two types of subspaces—a *scaling subspace*  $V_j$  and a *wavelet subspace*  $W_j$ , defined as follows (Bogess and Narcowich, 2001):

$$\begin{aligned} V_j &= \text{scaling subspace @ level } j \\ &= \text{signal approximation @ level } j = \sum_k a_{j,k} \phi_{j,k}(x), \end{aligned} \quad (3-7a)$$

$$\begin{aligned} W_j &= \text{wavelet subspace @ level } j = \text{signal details @ level } j \\ &= \sum_k b_{j,k} \psi_{j,k}(x), \end{aligned} \quad (3-7b)$$

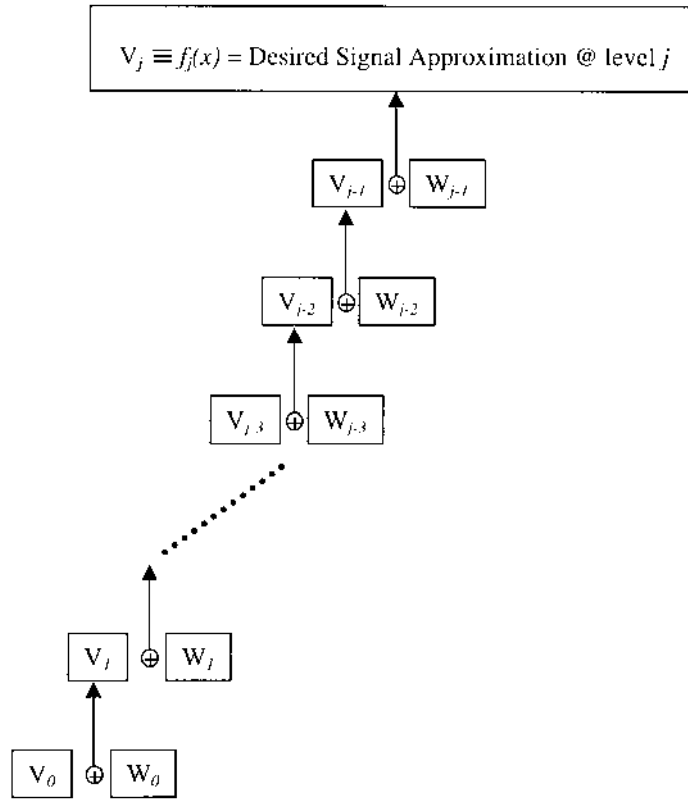
where  $a_{j,k}$  are the *scaling* coefficients and  $b_{j,k}$  are the *wavelet* coefficients at level  $j$ :

$$a_{j,k} = \langle f, \phi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \phi_{j,k}(x) dx, \quad (3-8a)$$

$$b_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \quad (3-8b)$$

In Eqs. (3-7a–b) and (3-8a–b), the level  $j$  corresponds to the wavelet (or scaling) level  $j$  as discussed in the previous section [see e.g., Eq. (2-6)]. Using this concept of subspaces within the MRA structure yields the following signal decomposition (see Fig. 1):

$$\begin{aligned} \text{Signal @ level } j &= f_j(x) = V_j \\ &= V_{j-1} \oplus W_{j-1} \\ &= V_{j-2} \oplus W_{j-2} \oplus W_{j-1} \\ &\vdots \\ &= V_0 \oplus W_0 \oplus W_1 \oplus \dots \oplus W_{j-1}. \end{aligned} \quad (3-9)$$



**Figure 1** Multi-resolution analysis tree.

The symbol  $\oplus$  indicates a *direct sum* in vector summation, where the  $V_j$ 's and  $W_j$ 's are  $N$ -length vectors in numerical terms. Hence using MRA makes it easy to represent a signal at any scaling/wavelet fine resolution level  $j$  by  $f_j = V_j$  or at a lower resolution level  $j-1$  by  $V_{j-1} \oplus W_{j-1}$ , and so forth.

The accuracy of this representation depends on both the smoothness of the function  $f(x)$  and on the digital filter coefficients. One measure of this accuracy is given by the *mean square error* (MSE)  $E_\Delta$ , which is defined by the relation

$$E_\Delta = ||f(x) - f_j(x)|| = \left[ \int_{-\infty}^{\infty} |f(x) - f_j(x)|^2 dx \right]^{\frac{1}{2}}. \quad (3-10)$$

The five important properties of MRA are listed below:

- 1)  $V_0 \subset V_1 \subset V_2 \subset \dots \subset V_j \subset V_{j+1} \subset \dots$
- 2)  $\cap V_j = \{0\}$ , and  $\cup V_j = -L^2(\mathfrak{R})$ ,  $\forall j \in \mathbb{Z}$
- 3)  $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
- 4)  $f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$
- 5)  $V_0$  has an orthonormal basis  $\{\phi(x-k) \mid \forall k \in \mathbb{Z}\}$ .

A particular requirement for the wavelet subspaces is that of *completeness*, which is the requirement that the MSE  $\rightarrow 0$  as  $j \rightarrow \infty$ , i.e.,  $f_j(x) \rightarrow f(x)$  as  $j \rightarrow \infty$ . Therefore completeness, combined with the subspace sequence given in Eq. (3-9), requires the following to hold true:

$$V_0 \oplus \sum_{j=0}^{\infty} W_j = L^2(\mathfrak{R}). \quad (3-11)$$

The requirement for completeness [Eq. (3-11)] can only be met if the transform functions (scaling and wavelet) form an *orthonormal* (O.N.) *basis* for the subspace decomposition (MRA). Therefore it is natural to require that  $\psi_{j,k}$  constitute an O.N. basis for  $W_j$  and  $\phi_{j,n}$  form an O.N. basis for  $V_j$  ( $\forall j, k, n \in \mathbb{Z}$ ). In order for  $\psi_{j,k}$  (or  $\phi_{j,n}$ ) to be an O.N. basis they must meet the following criteria (Bogess and Narcowich, 2001):

- i)  $\psi_{j,k}$  (or  $\phi_{j,n}$ ) must be O.N.
- ii) any  $f(x)$  in  $L^2(\mathfrak{R})$  can be approximated by a finite linear combination of  $\psi_{j,k}$ 's (or  $\phi_{j,n}$ 's).

Under the proper wavelet and scaling function generation, it can be shown that the orthogonality requirements for these functions are met. This may be stated as follows:

$$\begin{aligned} \text{i) } \phi(x-n) \text{ is orthogonal at all shifts } n: \\ \Rightarrow \int_{-\infty}^{\infty} \phi(x-n)\phi(x-m)dx = \delta(m-n). \end{aligned} \quad (3-12a)$$

ii)  $\phi(x-n)$  and  $\psi(x-n)$  are orthogonal to each other:

$$\Rightarrow \int_{-\infty}^{\infty} \phi(x-n)\psi(x-n)dx = 0. \quad (3-12b)$$

iii)  $\psi_{j,k}(x)$  is orthogonal at all scales and all shifts:

$$\Rightarrow \int_{-\infty}^{\infty} \psi_{j,k}(x)\psi_{m,n}(x)dx = \delta(j-m)\delta(k-n). \quad (3-12c)$$

These important concepts associated with digital filters, MRA, and O.N. are satisfied by the Symlet family of wavelets. The Symlet wavelet and general applications of wavelet transforms are discussed in the following section.

#### 4 APPLICATION OF WAVELET TRANSFORM TO THE NLS EQUATION

The NLS equation derived in Chapter 8 is restated below:

$$i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0. \quad (4-1)$$

In the MWR technique, the SYM6 scaling function series representation for the unknown function  $u$  becomes (Canuto et al., 1987)

$$u(\xi, \tau) = \sum_{l=0}^{N-1} a_l(\xi) \phi_l(\tau), \quad (4-2)$$

where the  $a_l(\xi)$   $\{l=0,1,\dots,N-1\}$  are the  $N$  unknown coefficients in  $\xi$  and  $\phi_l(\tau)$   $\{l=0,1,\dots,N-1\}$  are the  $N$  SYM6 scaling functions in  $\tau$ . The scaling functions are a function of  $\tau$  only and do not change with  $\xi$ . The scaling coefficients are a function of  $\xi$  and thus change as the pulse propagates along the fiber. For any given value of  $\xi$ , say  $\xi_0$ , these coefficients are calculated as follows:

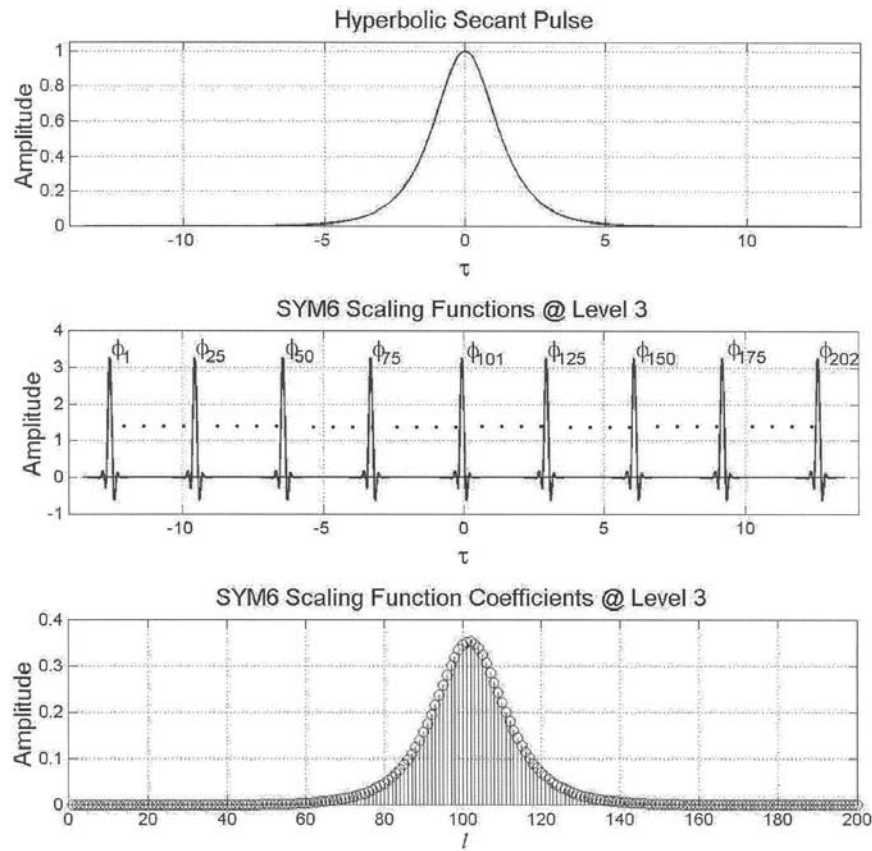
$$a_l(\xi_0) = \int_{-\infty}^{+\infty} u(\xi_0, \tau) \phi_l(\tau) d\tau; \quad \{l = 0, 1 \dots, N-1\} \quad (4-3)$$

Thus Eqs. (4-2) and (4-3) represent the *wavelet transform pair* as used in the *fully adaptive wavelet transform* (FAWT). As an example, Fig. B.2 shows a plot of a hyperbolic secant pulse as might be input to the FAWT, along with the level 3 scaling functions which are used in the transform and the associated level 3 scaling coefficients. Note that for the case of MRA at level 3, there are  $N=202$  scaling functions and correspondingly  $N=202$  scaling coefficients.

Applying the MWR to Eq. (B.4-1) will result in the following minimized residual equation:

$$\int_{\tau_{\min}}^{\tau_{\max}} \left[ i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u \right] \cdot \phi_k(\tau) d\tau = 0. \quad (4-4)$$

Note that Eq. (4-4) is the actual equation to be implemented in a numerical algorithm, so that the limits of integration  $[\tau_{\min}, \tau_{\max}]$  must have finite values. This will of course result in some numerical errors in the results. In essence, these limits of integration should be chosen as large as possible in order to reduce these numerical errors. However, the wider these limits are chosen, the more scaling functions which must be used (see Fig. 2), thus



**Figure 2** Hyperbolic secant pulse and associated transform coefficients at level 3.

increasing the value for  $N$ . If  $N$  becomes too large, then the size of the wavelet nonlinearity and self-steepening tensors derived below [ $\Phi$  and  $\Psi$ ], which are of size  $N^4$ , becomes unrealistic. Therefore the limits of integration  $N=202$ ,  $[\tau_{\min}, \tau_{\max}]$  and the number of scaling functions  $N$  must be carefully chosen in order to maintain numerical accuracy in an algorithm that has real-world applications. For the FAWT, these values were chosen to be  $N=202$ ,  $[\tau_{\min}, \tau_{\max}]=[-13.5, 13.5]$ . Note that in Fig. 2, each of the 202 scaling functions  $\phi_l(\tau)$  may be generated according to the dilation equation (3-1), or alternatively, using the MATLAB Wavelet Toolbox.

Expanding Eq. (4-4) term by term yields

$$\int i \frac{\partial u}{\partial \xi} \cdot \phi_k(\tau) d\tau + \int \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} \cdot \phi_k(\tau) d\tau + \int |u|^2 u \cdot \phi_k(\tau) d\tau = 0. \quad (4-5)$$

The three integro-differential terms on the left-hand side of Eq. (B.4-5) will be expanded and reduced as follows in order to derive a set of vector ordinary differential equations for the unknown scaling function coefficients  $a_l(\xi)$  (Stedham and Banerjee, 2000):

$$\begin{aligned} \int i \frac{\partial u}{\partial \xi} \cdot \phi_k(\tau) d\tau &= i \int \frac{\partial}{\partial \xi} \left[ \sum_l a_l(\xi) \phi_l(\tau) \right] \cdot \phi_k(\tau) d\tau \\ &= i \sum_l \frac{d}{d\xi} a_l(\xi) \cdot \int \phi_l(\tau) \phi_k(\tau) d\tau \\ &= i \frac{d}{d\xi} a(\xi) \cdot \mathbf{I} \quad [\mathbf{I} = \text{Identity Matrix}] \\ &= i a'(\xi), \end{aligned} \quad (4-6a)$$

$$\begin{aligned} \int \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} \cdot \phi_k(\tau) d\tau &= \frac{1}{2} \int \frac{\partial^2}{\partial \tau^2} \left[ \sum_l a_l(\xi) \phi_l(\tau) \right] \cdot \phi_k(\tau) d\tau \\ &= \frac{1}{2} \sum_l a_l(\xi) \cdot \int \frac{d^2 \phi_l(\tau)}{d\tau^2} \phi_k(\tau) d\tau \\ &= \frac{1}{2} a(\xi) \cdot \mathbf{K} \quad [\mathbf{K} = \text{Dispersion Matrix}], \end{aligned} \quad (4-6b)$$

$$\begin{aligned} \int |u|^2 u \cdot \phi_k(\tau) d\tau \\ = \int \left| \sum_l a_l(\xi) \phi_l(\tau) \right|^2 \left[ \sum_n a_n(\xi) \phi_n(\tau) \right] \cdot \phi_k(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
&= \int \left[ \sum_l a_l(\xi) \phi_l(\tau) \right] \left[ \sum_m a_m^*(\xi) \phi_m(\tau) \right] \left[ \sum_n a_n(\xi) \phi_n(\tau) \right] \cdot \phi_k(\tau) d\tau \\
&= \sum_l a_l(\xi) \cdot \left\{ \sum_n a_n(\xi) \cdot \left[ \int \phi_l(\tau) \phi_k(\tau) \phi_m(\tau) \phi_n(\tau) d\tau \right] \cdot \sum_m a_m^*(\xi) \right\} \\
&= \mathbf{a}(\xi) \cdot \{ \mathbf{a}(\xi) \cdot \Phi \cdot \mathbf{a}^*(\xi) \} \quad [\Phi = N^4 \text{ Kerr Nonlinearity Tensor}] \\
&= \mathbf{a}(\xi) \cdot \mathbf{C}(\xi) \quad [\mathbf{C}(\xi) = \text{Kerr Nonlinearity Matrix}].
\end{aligned} \tag{4-6c}$$

The orthogonality property of the SYM6 scaling functions [see Eq. (3-12a)] was used in converting the integral expression  $\int \phi_l(\tau) \phi_k(\tau) d\tau$  to the identity matrix  $\mathbf{I}$  in Eq. (4-6a). Combining the three terms derived in Eqs. (4-6a)–(4-6c) results in the following vector equation for updating the scaling function coefficients:

$$ia'(\xi) + \frac{1}{2} \mathbf{a}(\xi) \cdot \mathbf{K} + \mathbf{a}(\xi) \cdot \mathbf{C}(\xi) = 0. \tag{4-7}$$

In this equation  $\mathbf{a}(\xi)$  is a complex  $N$ -length vector which represents the  $N$  scaling coefficients that change as the algorithm is stepped in  $\xi$ . The  $N \times N$  real matrix  $\mathbf{K}$  is known as the linear dispersion matrix whose elements are derived from the scaling functions as follows:

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}, \quad \text{where } K_{lk} = \int \phi_l'' \phi_k d\tau. \tag{4-8}$$

The  $N \times N$  complex matrix  $\mathbf{C}(\xi)$  represents the nonlinear effects of the optical fiber. It is generated from a four-dimensional array (or tensor) which consists of an  $N \times N$  matrix whose “elements” are themselves  $N \times N$  matrices. The process by which  $\mathbf{C}(\xi)$  is created is portrayed below:

$$\mathbf{C}(\xi) = \begin{bmatrix} \mathbf{a}(\xi) \cdot \Phi_{11} \cdot \mathbf{a}^*(\xi) & \mathbf{a}(\xi) \cdot \Phi_{12} \cdot \mathbf{a}^*(\xi) & \cdots & \mathbf{a}(\xi) \cdot \Phi_{1N} \cdot \mathbf{a}^*(\xi) \\ \mathbf{a}(\xi) \cdot \Phi_{21} \cdot \mathbf{a}^*(\xi) & \mathbf{a}(\xi) \cdot \Phi_{22} \cdot \mathbf{a}^*(\xi) & \cdots & \mathbf{a}(\xi) \cdot \Phi_{2N} \cdot \mathbf{a}^*(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}(\xi) \cdot \Phi_{N1} \cdot \mathbf{a}^*(\xi) & \mathbf{a}(\xi) \cdot \Phi_{N2} \cdot \mathbf{a}^*(\xi) & \cdots & \mathbf{a}(\xi) \cdot \Phi_{NN} \cdot \mathbf{a}^*(\xi) \end{bmatrix}, \tag{4-9}$$

where each of the  $N \times N$  matrices  $\Phi_{lk}$  are real matrices with elements given by

$$\begin{aligned}
\Phi_{lk}(m, n) &= \int (\phi_l \phi_k) \cdot (\phi_m \phi_n) d\tau \quad \text{for } l, k, m, n \\
&= 0, 1, \dots, N-1.
\end{aligned} \tag{4-10}$$

Replacing the  $\xi$ -derivative term in Eq. (4-7) with a central difference approximation results in the following algorithm for updating the scaling coefficients in the wavelet domain:

$$\mathbf{a}(\xi + \Delta\xi) = \mathbf{a}(\xi) + i\Delta\xi \mathbf{a}(\xi) \cdot \left[ \frac{1}{2} \mathbf{K} + \mathbf{C}(\xi) \right]. \quad (4-11)$$

Therefore the procedure for computing pulse propagation using the FAWT is as follows:

- 1) For a given input pulse at an initial value of  $\xi_0$ , compute the initial set of scaling coefficients according to Eq. (4-3);
- 2) Update the scaling function coefficients using Eq. (4-11);
- 3) Compute the pulse propagation solution at any (using Eq. (4-2)).

Pulse propagation using the NLS is shown in Chapter 8, along with pulse propagation in media modeled by other nonlinear PDEs and solved using the FAWT technique.

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# Index

- (S)-(-)-N-(5-nitro-2-pyridyl)prolinol, 203
- 2-4-7 trinitro-9-flouride, 203
- 2k gratings, 199
- Absorptive bistability, 104
- Acceptor concentration, 177
- Acceptor to donor concentration, 187
- Acousto-optics, 145
- Admissibility criteria, 296
- Airy function, 183
- Airy pattern, 183
- Ammonium dihydrogen phosphate, 21
- Amplitude modulation, 22
- Amplitude modulator, 22
- Angular frequency, 6
- Angular plane wave spectrum, 127
- Anti-Stroke scattering, 151
- Bandgap, 271
- Barium titanate, 176, 181, 195, 199, 253
- Basic functions, 295
- Beam bending, 182
- Beam distortion, 181
- Beam fanning, 176, 181
  - deterministic, 181, 183
  - random, 181
- Beam propagation method, 290
- Beam propagation, 175
- Bessel functions, 157
- Binary phase gratings, 266
- Birefringence, 15, 210
  - wave plates, 16
- Bismuth silicon oxide, 264
- Bistability, 105, 111, 118, 272, 279, 282
  - absorptive, 101
  - dispersive, 101
- Bit rate-distance product, 155
- Bloch equations, 103
- Boundary conditions
  - hard, 226, 232
- Bragg angle, 145
- Bragg grating, 283
- Brillouin frequency, 147, 150
- Cascaded effective third-order nonlinearity, 97, 98
- Cat conjugator, 197

- Cavity
  - open, 255
- Characteristic impedance, 8, 9
- Cholesterol esters, 211
- Clearing temperature, 210
- Compact support, 293, 296, 298
- Compensating charge, 204
- Compensator, 17
- Completeness, 301
- Compressibility, 147
- Conjugate quadrature filter, 299
- Conservation of energy, 59
- Conservation of momentum, 59
- Conserved quantities, 281
- Constitutive relations, 156
- Continuity equation, 3
- Continuum theory, 213
- Contracted electro-optic coefficient, 181
- Contrapropagating pumps, 259
- Contrast enhancement, 264
- Coordinate system
  - cylindrical, 5
  - rectangular, 5
  - spherical, 5
- Coupled mode equations, 273
- Coupling coefficient
  - complex, 261
- Coupling constant, 258
  - complex, 260, 262
  - imaginary, 260
- Critical phase matching angle, 71
- Critical power, 86
- Cross-phase modulation, 50, 277
- Crystal
  - biaxial, 14
  - cubic, 14
  - negative uniaxial, 14
  - positive uniaxial, 14
  - uniaxial, 14, 20
- Cubic nonlinear coefficients, 55
- Cubic nonlinearity, 61, 276
- Current density, 177
- Degenerate four-wave mixing, 123
- Degrees of freedom, 244
- Density operator, 103
- Dielectric constant modulation, 264
- Dielectric modulation period, 265
- Dielectric tensor, 12, 14
- Diffraction, 25, 247
  - high-order, 180
- Diffraction efficiency, 251
- Diffraction-free beam, 89
- Diffraction optical elements, 265
- Diffusion, 177, 247, 261
  - constant, 177
  - length, 246
- Digital filters, 298
- Dipole moment density, 4
- Director distribution, 215, 217
- Dispersion, 31
  - anomalous, 159
  - normal, 159
  - topological, 277
  - waveguide, 32, 280
- Dispersion managed soliton system, 165
- Dispersion management, 164
- Dispersion relation, 272, 274, 275
- Displacement vector, 218
- Displays, 209
- Distributed feedback, 285
- Divergence theorem, 2
- Donor concentration, 177
- Doped polymer composite, 202
- Duffing's equation, 41
- Dyadic expression, 297
- Dyadic scale, 297
- Dynamical oscillation, 257
- Edge enhancement, 200, 202, 206, 254
- Edge-enhanced correlation, 206, 266
- Effective nonlinear refractive index
  - coefficient, 187, 228
  - liquid crystals, 240
- Effective third-order Raman susceptibility, 152
- Einstein convention, 14, 43

- Elastic constants
  - bend, 214
  - splay, 214
  - twist, 214
- Elastic free energy, 218
- Electric field, 218
  - strength, 1
- Electric flux density, 1
- Electrical energy density, 218
- Electron density, 179
- Electro-optic coefficients of crystals, 21
- Electro-optic effect, 20
- Electro-optic effect in uniaxial crystals, 20
- Electrostriction, 146
- Electrostrictive constant, 147
- Elliptic function, 73
- Ellipticity from  $z$ -scan, 193
- Energy level diagram, 151
- Energy transfer, 196
  
- Fabry–Perot cavity, 107, 255, 285
  - finesse, 255
- Fabry–Perot modes, 254, 255
- Fanning noise, 260
- Faraday’s law of induction, 2
- Feedback mirror, 245, 257
- Fiber-optic communication, 155, 164
- Filter
  - finite impulse response, 298
  - high-pass, 298
  - low-pass, 298
- Finite difference method, 233, 289
- Finite-energy functions, 296
- Floquet–Bloch functions, 274
- Forced nonlinear response of oscillator, 40
- Fourier analysis, 295
- Fourier optics, 30
- Fourier series, 273, 283
- Fourier transform, 26, 236
- Four-wave mixing, 123, 125
  - collinear, 128
  - degenerate, 197
- [Four-wave mixing]
  - noncollinear, 127, 129
  - nondegenerate, 197
  - pulsed, 128
  - pulsed pump, 129
- Fraunhofer approximation, 30
  - diffraction approximation, 30
  - diffraction formula, 30
- Freedericksz transition threshold, 210, 215
- Fresnel diffraction, 25, 28
  - formula, 28, 237
  - pattern, 30
- Fresnel equations, 113
- Fresnel reflection, 273
- Fresnel rings, 254
- Fully adaptive wavelet transform, 302, 306
- Fundamental, 60
  
- Gap soliton, 272, 277
- Gauss’ law, 2
- Gaussian beam, 28, 90, 91, 96, 176, 182, 236
  - elliptic, 188
  - periodic self-focusing, 82
  - self-focusing, 82, 86
  - waist size, 29
  - width, 29
- Gaussian pulse, 163
- Gibbs free energy, 224
- Glass transition temperature, 203
- Goos–Hanchen shift, 111
- Gordon–Haus effect, 164
- Graded index fiber, 89, 287
- Grating, 173, 287
  - reflection, 174, 248, 253, 271, 283
  - transmission, 174, 248, 253
- Grating number, 271
  
- Half-wave plate, 12, 24
- Half-wave voltage, 24
- Hamiltonian, 103
- Helical distortion, 209
- Helmholtz equation, 202

- Hexagonal cells, 241
- Hexagon formation, 258
  - time evolution, 258
- Hexagonal pattern, 242, 243, 248
- Hexagonal rotation, 251, 264
- Hexagonal sampling, of images, 264
- Hexagonal spot array, 250
- Hexagonal spots, 248
- Higher order diffraction, 201, 203
- Hologram
  - thin, 201
- Holographic current, 267
- Holographic subharmonic, 264
- Holography, 124
- Hydrodynamics, 243
- Hyperbolic secant pulse, 303
- Hysteresis, 101, 105, 109, 111, 118, 272, 279, 282
- Image broadcasting, 266
- Index ellipsoid, 17
- Indium tin oxide, 212
- Induced dielectric modulation, 179
- Induced focal length, 83, 92, 188
- Induced nonlinear refractive index, 184, 232
- Induced optical channels, 264
- Induced optical reorientation, 233
- Induced optical waveguides, 264
- Induced permittivity modulation, 180
- Induced reflection grating, 149, 195, 256
- Induced transmission grating, 153
- Information processing in nonlinear optics, 245
- Instability criterion, 257
- Intensity, 10
- Intensity-dependent refractive index, 50
- Intensity-dependent time constant, 180
- Intrinsic impedance, 8, 9
- Ionization cross-section, 177
- Ionized donor, 177
- Irradiance, 10
- Isotropic phase, 209
- Jacobian elliptic function, 281
- Jitter, 166
- Jones matrix, 12
- Jones vector, 12
- Kerr effect, 20, 50
- Kerr medium, 124, 245
- Kerr mirror, 257
- Kerr nonlinearity matrix, 305
- Kerr slice, 90, 245
- Kinematic wave equation, 32
- Korteweg deVries equation, 32
- $k$ -space formalism, 125
- Kukhtarev equation 175, 177
- Lagrange's equation, 214
- Laplace transform, 138
- Laplacian, 5
- Leap-frog technique, 291
- Lenz's law, 3
- Linear electro-optic coefficients, 20
- Linear electro-optic effect, 42, 48, 181
- Linear free-space propagation, 291
- Linear nondispersive–nonlinear dispersive interface, 112
- Linear–nonlinear interface, 101
  - grazing incidence, 110
- Linear optics, 38
- Linear polarization
  - frequency domain, 45
  - phasor, 45
  - time domain, 45
- Linear wave propagation
  - traveling wave solutions, 4
- Liquid crystal
  - chiral, 210
  - chloresteric, 210
  - director axis, 213
  - director distribution, 215
  - homeotropic, 213
  - homogeneous, 212

- [Liquid crystal]
  - homogeneously aligned, 224
  - induced molecular reorientation, 224
  - nematic, 210, 213
  - nonlinear optical properties, 224
  - smectic, 211
  - twist, 213
  - z-scan, 235
- Liquid crystal alignment
  - homeotropic, 211
  - homogeneous, 211
  - twist, 211
- Liquid crystal light valve, 245, 249
- Liquid crystalline phase, 210
- Liquid crystals, 209
  - nematic, 209
- Lithium niobate, 21, 176, 181, 187
- Longitudinal modulation, 22, 23
- Magnetic field strength, 1
- Magnetic flux density, 1
- Maxwell–Bloch equations, 102
- Maxwell relaxation time, 250
- Maxwell’s equations, 1, 4, 8, 156, 175
- Mean free limit, 105
- Mean square error, 301
- Medium
  - anisotropic, 12
  - dispersive, 32
  - homogeneous, 3
  - isotropic, 4
  - linear, 3
  - nondispersive, 32
- Mesogen
  - lyotropic, 210
  - thermotropic, 210
- Mesomorphic phase, 210
- Mesophase, 210
  - lyotropic, 210
  - thermotropic, 210
- Misalignment detection, 266
- Mobility
  - effective quasi-static, 177
- Molecular fields, 214
- Multimodal solution, 89
- Multi-resolution analysis, 298, 301
- Multi-resolution analysis tree, 300
- Nematic phase, 209
- N-ethyl carbazole, 203
- Newton’s iteration, 233
- Noise, 244
- Non-Bragg orders, 205
- Nondispersive medium, 6
- Nonlinear absorption coefficient, 96
- Nonlinear eigenmodes, 244, 260
  - in steady state, 259
- Nonlinear eikonal equations, 89
- Nonlinearity
  - inhomogeneous, 189
- Nonlinear Klein–Gordon equation, 114, 280
- Nonlinear model
  - alternate approach, 54
- Nonlinear optics
  - with feedback, 245
- Nonlinear refractive index coefficient, 79, 80, 160
- Nonlinear ring cavity, 101
- Nonlinear Schrodinger equation, 34, 72, 87, 115, 156, 160, 161, 279, 302
  - normalized, 162
- Nonlinear systems with feedback, 245
- Nonlinear wave equation, 56
- Nonspreading solution, 89
- Object beam, 196
- Object wave, 125
- On-axis transmittance, 93
- $oo-e$ - interaction, 61, 70
- Operator
  - linear differential, 290
  - nonlinear, 290
- Optical bistability, 42, 50, 61, 101, 209
  - photorefractive, 119
  - two-photon, 119

- Optical fiber
  - dispersion, 158
  - management, 164
  - slope, 158
  - graded-index, 156
  - group velocity, 158
  - linear, 156
  - mode propagation constant, 158
  - modes, 156
  - nonlinear, 155, 158
  - soliton, 161
  - step-index, 156
  - zero dispersion wavelength, 159
- Optical kaleidoscopes, 244
- Optically induced reorientational nonlinearity, 228
- Optical nonlinearity, 37
  - frequency domain, 44
  - modeling, 43
  - time domain, 43
- Optical phase conjugation, 123
- Optical shock formation, 171
- Optical soliton communication system, 166
- Optical spirals, 244
- Optical switching, 112
- Organic photorefractive materials, 202
- Orthonormal basis, 301
- Oscillator model of an atom, 38
- Oseen–Frank equation, 213
  
- Paraxial wave equation, 26, 163
- Pattern dynamics, 256
- Pattern hopping, 244, 254
- Period doubling, 264
- Periodic focusing, 81
- Periodic poling, 76, 283
- Permeability, 3
- Permittivity, 3
  - effective quasi-static, 177
- Permittivity modulation, 273
- Phase conjugate, 201
  - Gaussian pulse, 141
- [Phase conjugate]
  - reflectivity, 142
  - spatial variation, 136
- Phase conjugate transfer function, 139
- Phase conjugation, 42, 50, 61, 196, 209
  - impulse response, 139
  - $k$ -space formalism, 124
  - pulses, 137
  - transient response, 137
- Phase conjugator, 250
- Phase curvature, 29
- Phase distortion suppression, 245, 249
- Phase matching, 282
- Phase mismatch, 282
- Phase mismatched case, 61
- Photogalvanic current, 250
- Photogalvanic instability, 264
- Photoinduced holographic scattering, 249
- Photoinduced scattering, 245
- Photonic bandgap, 274
- Photonic bandgap structures, 76, 272
  - cubic nonlinearity, 276
  - higher-dimensional, 287
  - second harmonic generation, 282
  - soliton solutions, 277
- Photonic crystals, 272
- Photonic stop gap, 274
- Photorefractive effect, 175
- Photorefractive material, 149, 245
  - figure-of-merit, 202
- Photorefractive polymers, 176
- Photorefractive response, 180
- Photosensitive center, 204
- Photovoltaic coefficient, 187
- Photovoltaic effect, 184, 261
- Plane of polarization, 10
- Plane wave, 7
- Plane-wave propagation
  - uniaxial crystals, 15
- Plasticizing agent, 203
- Pockels coefficients, 20
- Pockels effect, 20, 42, 48
- Polarization, 10, 39, 43
  - atomic, 31

- [Polarization]
  - circular, 11
  - electronic, 31
  - elliptical, 11
  - ionic, 31
  - linear, 10
  - orientational, 31
  - second order, 46, 49
  - third order, 49, 80
- Polarization axis, 22
- Poly(N-vinylcarbazole), 203
- Potassium dihydrogen phosphate, 21
- Potassium niobate, 200, 245, 250, 251, 253, 256, 265
- Poynting vector, 8, 9
- Principal axis, 14, 23
- Principal dielectric constants, 14
- Probe, 123
  - spatial variation, 136
- Projection operator, 103
- Propagation constant, 6
  - effective, 114
- Propagation vector, 7
- Pseudospectral method, 289, 290
- Pump, 123
  - depleted, 282
  - undepleted, 282, 284
- $q$  parameter, 29, 82
- $q$  transformation, 187, 189
- $q$  transformation in nonlinear material, 83
- Quadratic nonlinear coefficient, 55
- Quarter-wave plate, 24
- Quartz, 21
- Quasi-linear limit, 72
- Quasi-phase matching, 61
- Radius of curvature, 29
- Raman gain, 162
  - response, 162
- Raman-Nath diffraction, 203
- Rate equations for scattering ring, 262
- Rayleigh-Bernard instability, 243
- Rayleigh length, 29, 91
- Rayleigh range, 30, 187, 190
- Real-time holography, 196
- Recombination coefficient, 204
- Rectangular waveguide, 274
- Reference beam, 196
- Reference wave, 125
- Reflection grating, 197, 260
- Reflectivity, 276
- Refractive index
  - extraordinary, 17
  - induced, 177
  - ordinary, 17
- Regularity, 297
- Resonance frequency, 40
  - nonlinearly modified, 40
- Resonant triad wavevector diagram, 60
- Retardation, 24
  - plate, 17
- Rigorous coupled wave analysis, 176
- Ring cavity, 107
  - Kerr type, 107
- Ring resonator, 256
- Rotary waves, 244
- Scaling coefficient, 299, 302
- Scaling functions, 296, 297
- Scaling operation, 295
- Scattering angle, 255
- Scattering cone, 251
- Scattering ratio, 262
- Scattering ring, 251, 262
- Schmitt trigger, 101
- Screening length, 250
- Second harmonic, 60
- Second harmonic enhancement, 286
- Second harmonic generation, 59, 62, 272, 282
  - in crystals, 62
  - d.c. induced, 50
  - depleted pump, 64
  - nonlinear transverse effects, 71

- [Second harmonic generation]
  - perfect phase matching, 66
  - perfect phase matching in
    - anisotropic crystal, 70
  - phase mismatched case, 67
  - quadratic and cubic nonlinearities, 73
  - undepleted fundamental, 63
- Second order polarization
  - frequency domain, 49
  - phasor, 49
  - time domain, 49
- Self-bending, 42, 79, 96
- Self-defocusing, 61, 79
- Self-focusing, 42, 50, 61, 79, 86, 156, 292
  - channel, 111
- Self-organization, 243, 245
  - behavioral patterns, 244
  - nonlinear optics, 244
  - patterns, 244
  - in photorefractive materials, 249
- Self-oscillation, 209
- Self-phase conjugation, 149, 200, 245, 254, 266
- Self-phase modulation, 42, 79, 155, 160, 277
- Self-pumped phase conjugate mirror, 253
- Self-pumped phase conjugation, 198, 199
- Self-refraction, 81
- Self-steepening, 156, 161, 171
  - equation, 172
- Self-trapped filament, 109
- Self-trapping bistability, 109
- Shaping factor, 255
- Shifting functions, 297
- Shock, 171
- Signum function, 32, 162
- Single beam holography, 191
- Six-wave mixing, 254
- Soliton, 87
  - bright, 166
  - dark, 166
  - dark pseudorandom, 171
- [Soliton]
  - fundamental, 163
  - gap, 272, 277
  - grating self-transparency, 278
  - optical, 155, 156
  - sech type, 88
  - spatial, 61, 88, 155
  - temporal, 61, 155
- Soliton propagation, 42
- Space charge field, 175, 177
- Space charge instability, 264
- Spatial dispersion curves, 256
- Spatial dispersion relation, 257, 258
- Spatial filtering, 202
- Spatial frequency, 184
- Spatial impulse response, 27
- Spatial sidebands, 259
- Spatial transfer function, 25
  - of propagation, 27
- Spatio-temporal pattern generation, 247
- Spectral broadening, 161
- Split-step beam propagation method, 184, 236, 290
- Square-integrable, 296
- Steady state eigenmode, 262
- Stimulated backscattering, 197
- Stimulated Brillouin scattering, 123, 145, 195, 197
  - gain factor, 149
- Stimulated photon emission, 146
- Stimulated photorefractive
  - backscattering, 198
- Stimulated Raman scattering, 151
  - gain coefficient, 152
- Stokes scattering, 151
- Stokes wave, 145
- Stop band, 271
- Strontium barium niobate, 181
- Subharmonic generation, 66
- Subspace
  - scaling, 299
  - wavelet, 299
- Surfactant, 213
- Susceptibility, 39, 41
  - cubic, 50

- [Susceptibility]
  - effective first order, 42
  - linear, 41
  - permutation symmetry, 48
  - second order, 41
  - third order, 41
- Susceptibility tensor, 43
  - linear, 45
  - quadratic, 46
- Switching, 42
- Symlet, 302
- Symmetrized split-step method, 291
- Symmetry, 297
- Symmetry breaking, 257
- Synergistic computing, 243
- Talbot effect, 250, 253
- Talbot imaging, 255
- Thermal excitation rate, 177
- Thermal generation rate, 204
- Thermotropic mesogen
  - chlolesteric, 210
  - nematic, 210
  - smectic, 210
- Thick sample, 187
- Thin sample, 183
- Third harmonic generation, 50
- Third order polarization, 51
  - frequency domain, 51
  - phasor, 51
  - time domain, 51
- Thirring model, 277, 278
- Threshold, 105, 112
  - condition, 257
  - gain, 260
- Tilt angle profile, 222
- Time constant, 246
- Topological dispersion, 283
- Torques, 214
  - bend, 214
  - electrical, 214
  - splay, 214
  - twist, 214
- Transition point, 210
- Transmission grating, 196
- Transmission resonance 285
- Transmittivity, 280
- Transverse electrical instability, 264
- Transverse mode pattern, 257
- Transverse modulation, 22
- Transverse optical bistability, 109
- Two-beam coupling, 175
- Two-wave coupling, 203
  - coefficient, 205
- Two-wave mixing, 194
- Type I interaction, 61
- Vanishing moments, 295
- Velocity, 6
- Vortex formation, 255
- Walkoff, 71
- Wave equation
  - homogeneous, 5
- Wave propagation, 12
- Wavefront, 7
  - reconstruction, 124
- Wavelet, 295
  - basis functions, 297
  - Coiflet, 299
  - Daubechies, 299
  - frequency-adapted, 295
  - Haar, 299
  - level, 299
  - Mexican hat, 299
  - Meyer, 299
  - Morlet, 299
  - mother, 295
  - symlet, 299
- Wavelet coefficient, 299
- Wavelet properties, 296
- Wavelet transform, 296
  - discrete, 297
  - full adaptive, 164, 172
- Winner-take-all route, 248
- Zero-dispersion wavelength, 162
- z-scan, 90, 91, 189, 250
  - liquid crystals, 235, 237
  - photorefractives, 191